LETTER FROM THE EDITOR

If you are on winter break and/or need to keep your mind working over the holidays, then you are in luck. The issue starts with Scott Chapman and Chris O'Neill's look at the Chicken McNugget problem (which is related to the Frobenius problem) that asks how many Chicken McNuggets can be bought in a single purchase if they are available only in packs of different numbers. They introduce the problem and relate it to questions motivated from the theory of nonunique factorization.

The next two articles revisit two calculus standards. Many picture triangles when they think of sines and cosines. Perhaps you are like me and think of the unit circle. Garth Isaak uses the geometry of the unit circle to provide an alternate proof of the derivative of the sine function. In their article, Jordan Bell and Viktor Blåsjö consider Pietro Mengoli's 1650 proof that the harmonic series diverges.

Has your well-worn oval track puzzle come apart? David Nash and Sara Randall answer the question: If the tiles are removed, how must they be returned in order to ensure that the puzzle is still solvable? To answer this question, they describe the groups representing arrangements of the tiles for all possible oval track puzzles.

It seems appropriate to include a Putnam exam-related article in the month of the Putnam exam. Thomas Koshy and Zhenguang Gao consider a polynomial extension of a difference equation problem from the 2007 Putnam exam.

We next have two proofs without words. First, Ángel Plaza uses a circle to show that the tangent plus cotangent is greater than or equal to 2. Then, Roger Nelsen computes the area of a square in two different ways to evaluate some periodic continued fractions. Some exercises are included.

Including a mixture of history, analysis, and combinatorics, Ádám Besenyei recalls a center-of-gravity idea from Picard to provide a mechanical interpretation of a proof of Chebyshev's sum inequality using a rearrangement inequality.

We next have a geometry interlude. Cherng-tiao Perng and Boyd Coan show how Euclid's principle can be used to show invariance of some areas. Zsolt Lengvarszky uses the sum and difference formulas for the tangent function to prove the Pythagorean theorem. And, Grégoire Nicollier, Aljoša Peperko, and Janez Šter provide perhaps the simplest dissection proof of the Pythagorean theorem. In the same spirit, as a boxed item later in the issue, Li Zhou provides a short proof of how the Law of Cosines implies the Law of Sines.

Douglass Grant uses linear algebra to prove Euler's four-square lemma that states that every integer can be written as the sum of four squares. Grant's article was the motivation for Dave Reimann's cover art for this issue that graphically shows 91 (the current volume number of the MAGAZINE) in the five ways that it can be written as the sum of four squares.

As with every issue in 2018, Lai Van Duc Thinh provides another Partiti puzzle (and its solution). However, in this issue, Stan Wagon explains how Integer-Linear Programming can be used to solve Partiti puzzles. A Mathematica file to do so is available as a supplement to his explanation.

With the Joint Math Meetings on the horizon, Brendan Sullivan provides a meeting-inspired crossword. Of course, there are problems to solve from the Problems section and some advice on winter break reading in the Reviews. Finally, we provide a list of recent referees and thank them for their service. Without referees, the MAA journals would not exist. Thank you!

Michael A. Jones, Editor

ARTICLES

Factoring in the Chicken McNugget Monoid

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People just want more of it.—Ray Kroc [9]

Every day, 34 million Chicken McNuggets are sold worldwide [4]. At most McDonalds locations in the United States today, Chicken McNuggets are sold in packs of 4, 6, 10, 20, 40, and 50 pieces. However, shortly after their introduction in 1979, they were sold in packs of 6, 9, and 20. The following problem spawned from the use of these latter three numbers.

The Chicken McNugget Problem. What numbers of Chicken McNuggets can be ordered using only packs with 6, 9, or 20 pieces?

Early references to this problem can be found in [28] and [32]. Positive integers satisfying the Chicken McNugget Problem are now known as *McNugget numbers* [23]. In particular, if n is a McNugget number, then there is an ordered triple $(a, b,$ and $c)$ of nonnegative integers such that

$$6a + 9b + 20c = n. \qquad (1)$$

We will call (a, b, c) a *McNugget expansion* of n (again see [23]). Since both (3, 0, 0) and (0, 2, 0) are McNugget expansions of 18, it is clear that McNugget expansions are not unique. This phenomenon will be the central focus of the remainder of this article.

If $\max\{a, b, c\} \geq 8$ in equation (1), then $n \geq 48$ and hence determining the numbers x with $0 \leq x \leq 48$ that are McNugget numbers can be checked either by hand or your favorite computer algebra system. The only such x's that are not McNugget numbers are: 1, 2, 3, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 22, 23, 25, 28, 31, 34, 37, and 43. (The non-McNugget numbers are sequence A065003 in the On-Line Encyclopedia of Integer Sequences [24].) We demonstrate this in Table 1 with a chart that offers the McNugget expansions (when they exist) of all numbers less than or equal to 50.

What happens with larger values? Table 1 has already verified that 44, 45, 46, 47, 48, and 49 are McNugget numbers. Hence, we have a sequence of six consecutive McNugget numbers, and by repeatedly adding 6 to these values, we obtain the following.

Proposition 1. *Any $x > 43$ is a McNugget number.*

Math. Mag. **91** (2018) 323–336. doi:10.1080/0025570X.2018.1515559
MSC: Primary 20M14, Secondary 11A51; 11B75; 20M13
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/umma.

#	(a, b, c)	#	(a, b, c)	#	(a, b, c)
0	(0, 0, 0)	17	NONE	34	NONE
1	NONE	**18**	(3, 0, 0) (0, 2, 0)	**35**	(1, 1, 1)
2	NONE	19	NONE	**36**	(0, 4, 0) (3, 2, 0) (6, 0, 0)
3	NONE	**20**	(0, 0, 1)	37	NONE
4	NONE	**21**	(2, 1, 0)	**38**	(0, 2, 1) (3, 0, 1)
5	NONE	22	NONE	**39**	(2, 3, 0) (5, 1, 0)
6	(1, 0, 0)	23	NONE	**40**	(0, 0, 2)
7	NONE	**24**	(4, 0, 0) (1, 2, 0)	**41**	(2, 1, 1)
8	NONE	25	NONE	**42**	(1, 4, 0) (4, 2, 0) (7, 0, 0)
9	(0, 1, 0)	**26**	(1, 0, 1)	43	NONE
10	NONE	**27**	(0, 3, 0) (3, 1, 0)	**44**	(1, 2, 1) (4, 0, 1)
11	NONE	28	NONE	**45**	(0, 5, 0) (3, 3, 0) (6, 1, 0)
12	(2, 0, 0)	**29**	(0, 1, 1)	**46**	(1, 0, 2)
13	NONE	**30**	(5, 0, 0) (2, 2, 0)	**47**	(0, 3, 1) (3, 1, 1)
14	NONE	31	NONE	**48**	(2, 4, 0) (5, 2, 0) (8, 0, 0)
15	(1, 1, 0)	**32**	(2, 0, 1)	**49**	(0, 1, 2)
16	NONE	**33**	(1, 3, 0) (4, 1, 0)	**50**	(2, 2, 1) (5, 0, 1)

TABLE 1: The McNugget numbers and their expansions from 0 to 50 with McNugget numbers listed in boldface.

Thus, 43 is the largest number of McNuggets that cannot be ordered using packs of size 6, 9, and 20.

Our aim in this article is to consider questions related to the multiple occurrences of McNugget expansions as seen in Table 1. Such investigations fall under the more general purview of the theory of non-unique factorizations in integral domains and monoids (a good technical reference on this subject is [21]). Using a general context, we show that the McNugget numbers form an additive monoid and discuss some properties shared by the class of additive submonoids of the nonnegative integers. We then define several combinatorial characteristics arising in non-unique factorization theory, and compute their explicit values for the McNugget monoid.

By emphasizing results concerning McNugget numbers, we offer the reader a glimpse into the vast literature surrounding non-unique factorizations. While we stick to the calculation of basic factorization invariants, our results indicate that such computations involve a fair amount of complexity. Many of the results we touch on have appeared in articles authored or coauthored by undergraduates in National Science Foundation sponsored REU programs. This is an area that remains rich in open problems, and we hope our discussion here spurs our readers (both young and old) to explore this rewarding subject further.

A brief diversion into generality

As illustrated above, Chicken McNugget numbers fit into a long studied mathematical concept. Variations on the Chicken McNugget Problem are known as the Postage Stamp Problem [27], the Coin Problem [16], and the Knapsack Problem [22]. Their general ideas are as follows. Given a set of k objects with predetermined values $n_1, n_2, \ldots, n_k$, what possible values of n can be had from combinations of these objects? Thus, if a value of n can be obtained, then there is an ordered k-tuple of nonnegative integers $(x_1, \ldots, x_k)$ that satisfies the linear diophantine equation

$$n = x_1 n_1 + x_2 n_2 + \cdots + x_k n_k. \tag{2}$$

We view this in a more algebraic manner. Given integers $n_1, \ldots, n_k > 0$, set

$$\langle n_1, \ldots, n_k \rangle = \{x_1 n_1 + \cdots + x_k n_k \mid x_1, \ldots, x_k \in \mathbb{N}_0\}.$$

Notice that if s_1 and s_2 are in $\langle n_1, \ldots, n_k \rangle$, then $s_1 + s_2$ is also in $\langle n_1, \ldots, n_k \rangle$. Since $0 \in \langle n_1, \ldots, n_k \rangle$ and $+$ is an associative operation, the set $\langle n_1, \ldots, n_k \rangle$ under $+$ forms a *monoid*. Monoids of nonnegative integers under addition, like the one above, are known as *numerical monoids*, and $n_1, \ldots, n_k$ are called *generators*. We will call the numerical monoid $\langle 6, 9, 20 \rangle$ the *Chicken McNugget monoid*, and denote it by 🐔.

Since 🐔 consists of the same elements as those in $\langle 6, 9, 20, 27 \rangle$, it is clear that generating sets are not unique. Using elementary number theory, it is easy to argue that any numerical monoid $\langle n_1, \ldots, n_k \rangle$ does have a unique generating set with minimal cardinality obtained by eliminating those generators n_i that lie in the numerical monoid generated by $\{n_1, \ldots, n_k\} - \{n_i\}$. In this way, it is clear that $\{6, 9, 20\}$ is indeed the minimal generating set of 🐔 since each cannot be factored in terms of the rest. Unless otherwise stated, when dealing with a general numerical monoid $\langle n_1, \ldots, n_k \rangle$, we will assume without loss of generality that the given generating set $\{n_1, \ldots, n_k\}$ is minimal.

In view of this broader setting, the Chicken McNugget Problem can be generalized as follows.

The Numerical Monoid Problem. If $n_1, \ldots, n_k$ are positive integers, then which nonnegative integers lie in $\langle n_1, \ldots, n_k \rangle$?

Example 1. We have already determined above exactly which nonnegative integers are McNugget numbers. Suppose the Post Office issues stamps in denominations of 4, 7, and 10 cents. What values of postage can be placed on a letter (assuming that as many stamps as necessary can be placed on the envelope)? In particular, we are looking for the elements of $\langle 4, 7, 10 \rangle$. We can again use brute force to find all the solutions to

$$4a + 7b + 10c = n$$

and conclude that 1, 2, 3, 5, 6, 9, and 13 cannot be obtained. Since 14, 15, 16, and 17 can, all postage values larger than 13 are possible. □

Let us return to the largest number of McNuggets that cannot be ordered (namely 43) and the companion number 13 obtained in Example 1. The existence of these numbers is no accident. To see this in general, let $n_1, \ldots, n_k$ be a set of positive integers that are relatively prime. By elementary number theory, there is a set $y_1, \ldots, y_k$ of (possibly negative) integers such that

$$1 = y_1 n_1 + \cdots + y_k n_k.$$

By choosing an element $V = x_1 n_1 + \cdots + x_k n_k \in \langle n_1, \ldots, n_k \rangle$ with sufficiently large coefficients (for instance, if each $x_i \geq n_1 |y_i|$), we see $V + 1, \ldots, V + n_1$ all lie in $\langle n_1, \ldots, n_k \rangle$ as well. As such, any integer greater than V can be obtained in $\langle n_1, \ldots, n_k \rangle$ by adding copies of n_1.

This motivates the following definition.

Definition 1. If $n_1, \ldots, n_k$ are relatively prime positive integers, then the *Frobenius number* of $\langle n_1, \ldots, n_k \rangle$, denoted $F(\langle n_1, \ldots, n_k \rangle)$, is the largest positive integer n such that $n \notin \langle n_1, \ldots, n_k \rangle$.

We have already shown that $F($🐔$) = 43$ and $F(\langle 4, 7, 10 \rangle) = 13$. A famous result of Sylvester from 1884 [31] states that if a and b are relatively prime, then

$$F(\langle a, b \rangle) = ab - a - b;$$

a nice proof of this can be found in [7]. This is where the fun begins, as strictly speaking no closed formula exists for the Frobenius number of numerical monoids that require 3 or more generators. While there are fast algorithms that can compute $F(\langle n_1, n_2, n_3\rangle)$ (see for instance [18]), at best formulas for $F(\langle n_1, \ldots, n_k\rangle)$ exist only in special cases (you can find one such special case, where $F($$) = 43$ pops out in [1, p. 14]). Our purpose is not to compile or expand upon the vast literature behind the Frobenius number; in fact, we direct the reader to the excellent monograph of Ramírez Alfonsín [29] for more background reading on the Diophantine Frobenius Problem.

The McNugget factorization toolkit

We focus now on the multiple McNugget expansions we saw in Table 1. In particular, notice that there are McNugget numbers that have unique triples associated to them (6, 9, 12, 15, 20, 21, 26, 29, 32, 35, 40, 41, 46, and 49), some that have two (18, 24, 27, 30, 33, 35, 39, 44, 47, and 50), and even some that have three (36, 42, 45, and 48). While the "normal" notion of factoring occurs in systems where multiplication prevails, notice that the ordered triples representing McNugget numbers are actually *factorizations* of these numbers into "additive" factors of 6, 9, and 20.

Let's borrow some terminology from abstract algebra ([19] is a good beginning reference on the topic). Let x and $y \in \langle n_1, \ldots, n_k\rangle$. We say that x *divides* y in $\langle n_1, \ldots, n_k\rangle$ if there exists a $z \in \langle n_1, \ldots, n_k\rangle$ such that $y = x + z$. We call a nonzero element $x \in \langle n_1, \ldots, n_k\rangle$ *irreducible* if whenever $x = y + z$, either $y = 0$ or $z = 0$. (Hence, x is irreducible if its only proper divisors are 0 and itself). Both of these definitions are obtained from the usual "multiplicative" definition by replacing "·" with "+" and 1 with 0.

We leave the proof of the following to the reader.

Proposition 2. *If $\langle n_1, \ldots, n_k\rangle$ is a numerical monoid, then its irreducible elements are precisely $n_1, \ldots, n_k$.*

Related to irreducibility is the notion of prime elements. Borrowing again from the multiplicative setting, a nonzero element $x \in \langle n_1, \ldots, n_k\rangle$ is *prime* if whenever x divides a sum $y + z$, then either x divides y or x divides z. It is easy to check from the definitions that prime elements are always irreducible, but it turns out that in general irreducible elements need not be prime. In fact, the irreducible elements $n_1, \ldots, n_k$ of a numerical monoid are never prime. To see this, let n_i be an irreducible element and let T be the numerical monoid generated by $\{n_1, \ldots, n_k\} - \{n_i\}$. Although $n_i \notin T$, some multiple of n_i must lie in T (take, for instance, $n_2 n_i$). Let $kn = \sum_{j\neq i} x_j n_j$ (for some $k > 1$) be the smallest multiple of n_i in T. Then n divides $\sum_{j\neq i} x_j n_j$ over $\langle n_1, \ldots, n_k\rangle$, but by the minimality of k, n does not divide any proper subsum. Thus n_i is not prime.

For our purposes, we restate Proposition 2 in terms of .

Corollary 1. *The irreducible elements of the McNugget monoid are 6, 9, and 20. There are no prime elements.*

The set of factorizations of an element. We refer once again to the elements in Table 1 with multiple irreducible factorizations. For each $x \in$ , let

$$\mathsf{Z}(x) = \{(a, b, c) \mid 6a + 9b + 20c = x\}.$$

We will refer to $\mathsf{Z}(x)$ as the *complete set of factorizations* x in , and as such, we could relabel columns 2, 4, and 6 of Table 1 as "$\mathsf{Z}(x)$." While we will not dwell on general structure problems involving $\mathsf{Z}(x)$, we do briefly address one in the next example.

Example 2. What elements $x \in$ 🐔 are uniquely factorable (i.e., when is $|\mathsf{Z}(x)| = 1$)? A quick glance at Table 1 yields 14 such nonzero elements (namely 6, 9, 12, 15, 20, 21, 26, 29, 32, 35, 40, 41, 46, and 49). Are there others? We begin by noting in Table 1 that

$$(3, 0, 0), (0, 2, 0) \in \mathsf{Z}(18) \qquad \text{and} \qquad (10, 0, 0), (0, 0, 3) \in \mathsf{Z}(60).$$

This implies that in any factorization in 🐔, 3 copies of 6 can be freely replaced with 2 copies of 9 (this is called a *trade*). Similarly, 2 copies of 9 can be traded for 3 copies of 6, and 3 copies of 20 can be traded for 10 copies of 6. In particular, for any element $n = 6a + 9b + 20c \in$ 🐔, if $a \geq 3$, $b \geq 2$ or $c \geq 3$, then n has more than one factorization in 🐔. As such, if n is to have unique factorization, then $0 \leq a \leq 2$, $0 \leq b \leq 1$, and $0 \leq c \leq 2$. This leaves 18 possibilities, and a quick check yields that the 3 missing elements are $52 = (2, 0, 2)$, $55 = (1, 1, 2)$, and $61 = (2, 1, 2)$. □.

The argument in Example 2 easily generalizes—every numerical monoid that requires more than one generator has finitely many elements that factor uniquely—but note that minimal trades need not be as simple as replacing a multiple of one generator with a multiple of another. Indeed, in the numerical monoid $\langle 5, 7, 9, 11 \rangle$, there is a trade $(1, 0, 0, 1), (0, 1, 1, 0) \in \mathsf{Z}(16)$, though 16 is not a multiple of any generator. Determining the "minimal" trades of a numerical monoid, even computationally, is known to be a very hard problem in general [30].

The length set of an element and related invariants. Extracting information from the factorizations of numerical monoid elements (or even simply writing them all down) can be a tall order. To this end, combinatorially flavored *factorization invariants* are often used, assigning to each element (or to the monoid as a whole) a value measuring its failure to admit unique factorization. We devote the remainder of this article to examining several factorization invariants, and what they tell us about the McNugget monoid as compared to more general numerical monoids.

We begin by considering a set, derived from the set of factorizations, that has been the focus of many articles in the mathematical literature over the past 30 years. If $x \in$ 🐔 and $(a, b, c) \in \mathsf{Z}(x)$, then the *length* of the factorization (a, b, c) is denoted by

$$|(a, b, c)| = a + b + c.$$

We have shown earlier that factorizations in 🐔 may not be unique, and a quick look at Table 1 shows that their lengths can also differ. For instance, 42 has three different factorizations, with lengths 5, 6, and 7, respectively. Thus, we denote the *set of lengths* of x in 🐔 by

$$\mathcal{L}(x) = \{|(a, b, c)| : (a, b, c) \in \mathsf{Z}(x)\}.$$

In particular, $\mathcal{L}(42) = \{5, 6, 7\}$. Moreover, set

$$\ell(x) = \min \mathcal{L}(x) \qquad \text{and} \qquad L(x) = \max \mathcal{L}(x).$$

(In our setting, it is easy to argue that $\mathcal{L}(x)$ must be finite, so the maximum and minimum above are both well defined.) To give the reader a feel for these invariants, in Table 2 we list all the McNugget numbers from 1 to 50 and their associated values $\mathcal{L}(x)$, $\ell(x)$, and $L(x)$.

The following recent result describes the functions $L(x)$ and $\ell(x)$ for elements $x \in \langle n_1, \ldots, n_k \rangle$ that are *sufficiently large* with respect to the generators. Intuitively, Theorem 1 says that for "most" elements x, any factorization with maximal length is almost entirely comprised of n_1, so $L(x + n_1)$ is obtained by taking a maximum length factorization for x and adding one additional copy of n_1. In general, the

x	$\mathcal{L}(x)$	$\ell(x)$	$L(x)$	x	$\mathcal{L}(x)$	$\ell(x)$	$L(x)$	x	$\mathcal{L}(x)$	$\ell(x)$	$L(x)$
0	{0}	0	0	**27**	{3, 4}	3	4	**41**	{4}	4	4
6	{1}	1	1	**29**	{2}	2	2	**42**	{5, 6, 7}	5	7
9	{1}	1	1	**30**	{4, 5}	4	5	**44**	{4, 5}	4	5
12	{2}	2	2	**32**	{3}	3	3	**45**	{5, 6, 7}	5	7
15	{2}	2	2	**33**	{4, 5}	4	5	**46**	{3}	3	3
18	{2, 3}	2	3	**35**	{3}	3	3	**47**	{4, 5}	4	5
20	{1}	1	1	**36**	{4, 5, 6}	4	6	**48**	{6, 7, 8}	6	8
21	{3}	3	3	**38**	{3, 4}	3	4	**49**	{3}	3	3
24	{3, 4}	3	4	**39**	{5, 6}	5	6	**50**	{5, 6}	5	6
26	{2}	2	2	**40**	{2}	2	2				

TABLE 2: The McNugget numbers from 0 to 50 with $\mathcal{L}(x)$, $\ell(x)$, and $L(x)$.

"sufficiently large" hypothesis is needed, since, for example, both $41 = 2 \cdot 9 + 1 \cdot 23$ and $50 = 5 \cdot 10$ are maximum length factorizations in the numerical monoid $\langle 9, 10, 23 \rangle$.

Theorem 1 ([5, Theorems 4.2 and 4.3]). *Suppose $\langle n_1, \ldots, n_k \rangle$ is a numerical monoid. If $x > n_1 n_k$, then*

$$L(x + n_1) = L(x) + 1,$$

and if $x > n_{k-1} n_k$, then

$$\ell(x + n_k) = \ell(x) + 1.$$

We will return to this result later when we give a closed formula for $L(x)$ and $\ell(x)$ that holds for all $x \in$ 🐔.

Given our definitions to this point, we now mention perhaps the most heavily studied invariant in the theory of non-unique factorization. For $x \in \langle n_1, \ldots, n_k \rangle$, the ratio

$$\rho(x) = \frac{L(x)}{\ell(x)},$$

is called the *elasticity* of x, and

$$\rho(\langle n_1, \ldots, n_k \rangle) = \sup\{\rho(x) \mid x \in \langle n_1, \ldots, n_k \rangle\}$$

is the *elasticity* of $\langle n_1, \ldots, n_k \rangle$. The elasticity of an element $n \in \langle n_1, \ldots, n_k \rangle$ measures how "spread out" its factorization lengths are; the larger $\rho(n)$ is, the more spread out $\mathcal{L}(n)$ is. To this end, the elasticity $\rho(\langle n_1, \ldots, n_k \rangle)$ encodes the highest such "spread" throughout the entire monoid. For example, if $\rho(\langle n_1, \ldots, n_k \rangle) = 2$, then the maximum factorization length of any element $n \in \langle n_1, \ldots, n_k \rangle$ is at most twice its minimum factorization length.

A formula for the elasticity of a general numerical monoid, given below, was given in [12], and was the result of an undergraduate research project.

Theorem 2 ([12], Theorem 2.1 and Corollary 2.3). *The elasticity of the numerical monoid $\langle n_1, \ldots, n_k \rangle$ is*

$$\rho(\langle n_1, \ldots, n_k \rangle) = n_k / n_1.$$

Moreover, $\rho(n) = n_k/n_1$ precisely when n is an integer multiple of the least common multiple of n_1 and n_k, and for any rational $r < n_k/n_1$, there are only finitely many elements $x \in \langle n_1, \ldots, n_k \rangle$ with $\rho(x) \leq r$.

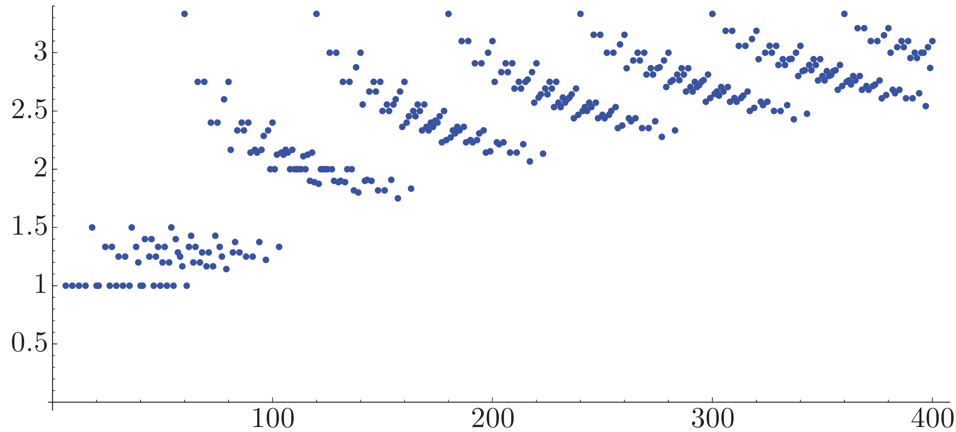

Figure 1 A plot depicting the elasticity function $\rho(n)$ for $n \in$ 🐔.

The significance of the final statement in Theorem 2 is that there are rational values $q \in [1, n_k/n_1]$ that do not lie in the set $\{\rho(x) \mid x \in \langle n_1, \ldots, n_k \rangle\}$ and hence

$$\{\rho(x) \mid x \in \langle n_1, \ldots, n_k \rangle\} \subsetneq \mathbb{Q} \cap [1, n_k/n_1];$$

to use terminology from the literature, numerical monoids are never *fully elastic*. Figure 1 depicts the elasticities of elements of 🐔 up to $n = 400$; indeed, as n increases, the elasticity $\rho(n)$ appears to converge to $10/3 = \rho($🐔$)$. In general, the complete image $\{\rho(x) \mid x \in \langle n_1, \ldots, n_k \rangle\}$ has been determined by Barron, O'Neill, and Pelayo in another student coauthored article [5, Corollary 4.5]; we direct the reader there for a thorough mathematical description of Figure 1.

We close our discussion of elasticity with the following.

Corollary 2. *The elasticity of the McNugget monoid is* $\rho($🐔$) = 10/3$.

While a popular invariant to study, the elasticity only tells us about the largest and smallest elements of $\mathcal{L}(x)$. Looking at Table 2, it appears that the length sets of the first few McNugget numbers are uniformly constructed (each is of the form $[a, b] \cap \mathbb{N}$ for positive integers a and b). One need not look too much further to break this pattern; the element $60 \in$ 🐔 has

$$\mathsf{Z}(60) = \{(0, 0, 3), (1, 6, 0), (4, 4, 0), (7, 2, 0), (10, 0, 0)\}$$

and thus $\mathcal{L}(60) = \{3, 7, 8, 9, 10\}$. This motivates the following "finer" factorization invariant. Fix $x \in \langle n_1, \ldots, n_k \rangle$, and let $\mathcal{L}(x) = \{m_1, \ldots, m_t\}$ with $m_1 < m_2 < \cdots < m_t$. Define the *delta set* of x as

$$\Delta(x) = \{m_i - m_{i-1} \mid 2 \leq i \leq t\},$$

and the *delta set* of $\langle n_1, \ldots, n_k \rangle$ as

$$\Delta(\langle n_1, \ldots, n_k \rangle) = \bigcup_{x \in \langle n_1, \ldots, n_k \rangle} \Delta(x).$$

The study of the delta sets of numerical monoids (and more generally, of cancellative commutative monoids) has been an extremely popular topic; many such articles feature results from REU programs (see, for instance, [8],[10], [11], [13], [14], and [15]).

From Table 1 we see that the McNugget numbers from 1 to 50 all have delta set $\emptyset$ or $\{1\}$, and we have further showed that $\Delta(60) = \{1, 4\}$. What is the delta set of 🐔

and moreover, what possible subsets of this set occur as $\Delta(x)$ for some $x \in$ 🐔? We will later address those questions with the help of a result from [13], stated below as Theorem 3.

One of the primary difficulties in determining the set $\Delta(\langle n_1, \ldots, n_k \rangle)$ is that even though $\Delta(x)$ is finite for every element x, the definition of $\Delta(\langle n_1, \ldots, n_k \rangle)$ involves the union of infinitely many such sets. The key turns out to be a description of the sequence $\{\Delta(x)\}_{x \in \langle n_1, \ldots, n_k \rangle}$ for large x (note that this is a sequence of sets, not integers). Paul Baginski conjectured during the writing of [8] that this sequence is eventually periodic, and three years later this was settled in the affirmative, again in an REU project.

Theorem 3 ([13, Theorem 1 and Corollary 3])**.** *For* $x \in \langle n_1, \ldots, n_k \rangle$,

$$\Delta(x) = \Delta(x + n_1 n_k),$$

whenever $x > 2kn_2n_k^2$. *In particular,*

$$\Delta(\langle n_1, \ldots, n_k \rangle) = \bigcup_{x \in D} \Delta(x),$$

where $D = \{x \in \langle n_1, \ldots, n_k \rangle \mid x \le 2kn_2n_k^2 + n_1 n_k\}$ *is a finite set.*

Thus, $\Delta(\langle n_1, \ldots, n_k \rangle)$ can be computed in finite time. The bound given in Theorem 3 is far from optimal; it is drastically improved in [20], albeit with a much less concise formula. For convenience, we will use the bound given above in our calculation of $\Delta($🐔$)$ in the next section.

Beyond the length set. We remarked earlier that no element of a numerical monoid is prime. Let us consider this more closely in 🐔. For instance, since 6 is not prime, there is a sum $x + y$ in 🐔 such that 6 divides $x + y$, but 6 does not divide x nor does 6 divide y (take, for instance, $x = y = 9$). But note that 6 satisfies the following slightly weaker property. Suppose that 6 divides a sum $x_1 + \cdots + x_t$ where $t > 3$. Then, there is a subsum of at most 3 of the x_i's that 6 does divide. To see this, notice that if 6 divides any of the x_i's, then we are done. So suppose it does not. If 9 divides both x_i and x_j, then 6 divides $x_i + x_j$ since 6 divides $9 + 9$. If no two x_i's are divisible by 9, then at least 3 x_i's are divisible by 20, and nearly identical reasoning to the previous case completes the argument. This value of 3 offers some measure as to how far 6 is from being prime, and motivates the following definition.

Definition 2. Let $\langle n_1, \ldots, n_k \rangle$ be a numerical monoid. Given any positive element $x \in \langle n_1, \ldots, n_k \rangle$, define $\omega(x) = m$ if m is the smallest positive integer such that whenever x divides $x_1 + \cdots + x_t$, with $x_i \in \langle n_1, \ldots, n_k \rangle$, then there is a subset $T \subset \{1, 2, \ldots, t\}$ of indices with $|T| \le m$ such that x divides $\sum_{i \in T} x_i$.

Using Definition 2, a prime element would have ω-value 1, so $\omega(x)$ can be interpreted as a measure of how far x is from being prime. In 🐔, we argued that $\omega(6) = 3$; a similar argument yields $\omega(9) = 3$ and $\omega(20) = 10$. Notice that the computation of $\omega(x)$ is dependent more on $\mathsf{Z}(x)$ than $\mathcal{L}(x)$, and hence encodes much different information than either $\rho(x)$ or $\Delta(x)$.

Let us examine more closely the argument that $\omega(6) = 3$. The key is that 6 divides $9 + 9$ and $20 + 20 + 20$, but does not divide any subsum of either. Indeed, the latter of these expressions yields a lower bound of $\omega(6) \ge 3$, and the given argument implies that equality holds. With this in mind, we give the following equivalent form of Definition 2, which often simplifies the computation of $\omega(x)$.

Theorem 4 ([26, Proposition 2.10]). *Suppose $\langle n_1, \ldots, n_k \rangle$ is a numerical monoid and $x \in \langle n_1, \ldots, n_k \rangle$. The following conditions are equivalent.*

(a) $\omega(x) = m$.

(b) *m is the maximum length of a sum $x_1 + \cdots + x_t$ of irreducible elements in $\langle n_1, \ldots, n_k \rangle$ with the property that (i) x divides $x_1 + \cdots + x_t$, and (ii) x does not divide $x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_t$ for $1 \le j \le t$.*

The sum $x_1 + \cdots + x_t$ alluded to in part (b) above is called a *bullet* for x. Hence, $20 + 20 + 20$ is a bullet for 6 in 🐔, and moreover has maximal length. The benefit of Theorem 4 is twofold: (i) each $x \in \langle n_1, \ldots, n_k \rangle$ has only finitely many bullets, and (ii) the list of bullets can be computed in a similar fashion to the set $\mathsf{Z}(x)$ of factorizations. We refer the reader to [3, 6], both of which give explicit algorithms (again resulting from undergraduate research projects) for computing ω-values.

Our goal is to completely describe the behavior of the ω-function of the McNugget monoid. We do so in Theorem 9 using the following result, which is clearly similar in spirit to Theorems 1 and 3.

Theorem 5 ([25, Theorem 3.6]). *For $x \in \langle n_1, \ldots, n_k \rangle$ sufficiently large,*

$$\omega(x + n_1) = \omega(x) + 1.$$

In particular, this holds for

$$x > \frac{F + n_2}{n_2/n_1 - 1},$$

where $F = F(\langle n_1, \ldots, n_k \rangle)$ is the Frobenius number.

The similarity between Theorems 1 and 5 is not a coincidence. While $L(x)$ and $\omega(x)$ are indeed different functions (for instance, $L(6) = 1$ while $\omega(6) = 3$), they are closely related; the ω-function can be expressed in terms of max factorization length that is computed when some collections of generators are omitted. We direct the interested reader to [6, Section 6], where an explicit formula of this form for $\omega(n)$ is given.

Calculations for the Chicken McNugget monoid

In the final section of this article, we give explicit expressions for $L(x)$, $\ell(x)$, $\Delta(x)$ and $\omega(x)$ for every $x \in$ 🐔. The derivation of each such expression makes use of a theoretical result presented earlier.

We note that each of the formulas provided in this section could also be derived in a purely computational manner, using Theorems 1, 3, and 5 and the inductive algorithms introduced in [6] (indeed, these computations finish in a reasonably short amount of time using the implementation in the `numericalsgps` package discussed in the appendix). However, several of the following results identify an interesting phenomenon that distinguish 🐔 from more general numerical monoids (see the discussion preceding the Question at the end of this article), and the arguments that follow give the reader an idea of how theorems involving factorization in numerical monoids can be proved.

Calculating factorization lengths. Theorem 1 states that $L(x + n_1) = L(x) + 1$ and $\ell(x + n_k) = \ell(x) + 1$ for sufficiently large $x \in \langle n_1, \ldots, n_k \rangle$. but, it was observed during the writing of [5] that for many numerical monoids, the "sufficiently large" requirement is unnecessary. As it turns out, one such example is the McNugget monoid 🐔, which we detail below.

Theorem 6. *For every element* $x \in$ 🐔, $L(x+6) = L(x) + 1$. *In particular, if we write* $x = 6q + r$ *for* $q, r \in \mathbb{N}$ *and* $r < 6$, *then, for each* $x \in$ 🐔,

$$L(x) = \begin{cases} q & \text{if } r = 0 \text{ or } 3, \\ q - 5 & \text{if } r = 1, \\ q - 2 & \text{if } r = 2 \text{ or } 5, \\ q - 4 & \text{if } r = 4. \end{cases}$$

Proof. Fix $x \in$ 🐔 and a factorization $(a, b,$ and $c)$ of x. If $b > 1$, then x has another factorization $(a+3, b-2, c)$ with length $a+b+c+1$. Similarly, if $c \geq 3$, then $(a + 10, b, c-3)$ is also a factorization of x and has length $a+b+c+7$. This implies that if (a, b, c) has maximum length among factorizations of x, then $b \leq 1$ and $c \leq 2$. Upon inspecting Table 1, we see that unless $x \in \{0, 9, 20, 29, 40, 49\}$, we must have $a > 0$.

Now, assume (a, b, c) has maximum length among factorizations of x. We claim $(a+1, b, c)$ is a factorization of $x+6$ with maximum length. From Table 1, we see that since $x \in$ 🐔, we must have $x + 6 \notin \{0, 9, 20, 29, 40, 49\}$, meaning any maximum length factorization of $x+6$ must have the form $(a'+1, b', c')$. This yields another factorization (a', b', c') of x, and since (a, b, c) has maximum length, we have $a + b + c \geq a' + b' + c'$. As such, $(a+1, b, c)$ is at least as long as $(a'+1, b', c')$, and the claim is proved. Thus,

$$L(x+6) = a + 1 + b + c = L(x) + 1.$$

From here, the given formula for $L(x)$ now follows from the first claim and the values $L(0)$, $L(9)$, $L(20)$, $L(29)$, $L(40)$, and $L(49)$ in Table 2. ■

A similar expression can be obtained for $\ell(x)$, ableit with 20 cases instead of 6, this time based on the value of x modulo 20. We encourage the reader to adapt the argument above for Theorem 7.

Theorem 7. *For every element* $x \in$ 🐔, $\ell(x+20) = \ell(x) + 1$. *In particular, if we write* $x = 20q + r$ *for* $q, r \in \mathbb{N}$ *and* $r < 20$, *then, for each* $x \in$ 🐔,

$$\ell(x) = \begin{cases} q & \text{if } r = 0, \\ q + 1 & \text{if } r = 6, 9, \\ q + 2 & \text{if } r = 1, 4, 7, 12, 15, 18, \\ q + 3 & \text{if } r = 2, 5, 10, 13, 16, \\ q + 4 & \text{if } r = 8, 11, 14, 19, \\ q + 5 & \text{if } r = 3, 17. \end{cases}$$

Theorems 6 and 7 together yield a closed form for $\rho(x)$ that holds for all $x \in$ 🐔. Since $\operatorname{lcm}(6, 20) = 60$ cases are required, we leave the construction of this closed form to the interested reader.

Calculating delta sets. Unlike maximum and minimum factorization length, $\Delta(x)$ is periodic for sufficiently large $x \in$ 🐔. For example, a computer algebra system can be used to check that $\Delta(91) = \{1\}$ while $\Delta(211) = \{1, 2\}$. Theorem 3 guarantees $\Delta(x + 120) = \Delta(x)$ for $x > 21600$, but some considerable reductions can be made. In particular, we will reduce the period from 120 down to 20, and will show that equality holds for all $x \geq 92$ (that is to say, 91 is the largest value of x for which $\Delta(x + 20) \neq \Delta(x)$).

Theorem 8. *Each* $x \in$ 🐔 *with* $x \geq 92$ *has* $\Delta(x+20) = \Delta(x)$. *Moreover,*

$$\Delta(x) = \begin{cases} \{1\} & \text{if } r = 3, 8, 14, 17, \\ \{1,2\} & \text{if } r = 2, 5, 10, 11, 16, 19, \\ \{1,3\} & \text{if } r = 1, 4, 7, 12, 13, 18, \\ \{1,4\} & \text{if } r = 0, 6, 9, 15, \end{cases}$$

where $x = 20q + r$ *for* $q, r \in \mathbb{N}$ *and* $r < 20$. *Hence* $\Delta($🐔$) = \{1, 2, 3, 4\}$.

Proof. We will show that $\Delta(x+20) = \Delta(x)$ for each $x > 103$. The remaining claims can be verified by extending Table 2 using computer software.

Suppose $x > 103$, fix a factorization (a, b, c) for x, and let $l = a + b + c$. If $c \geq 3$, then x also has $(a+10, b, c-3)$, $(a+7, b+2, c-3)$, $(a+4, b+4, c-3)$, and $(a+1, b+6, c-3)$ as factorizations, meaning

$$\{l, l+4, l+5, l+6, l+7\} \subset \mathcal{L}(x).$$

Alternatively, since $x > 103$, if $c \leq 2$, then $6a + 9b \geq 63$, and thus

$$l \geq a + b + 2 \geq 9 \geq \ell(x) + 4.$$

The above arguments imply (i) any gap in successive lengths in $\mathcal{L}(x)$ occurs between $\ell(x)$ and $\ell(x) + 4$, and (ii) every factorization with length in that interval has at least one copy of 20. As such, $x + 20$ has the same gaps below $\ell(x+20) + 4$ as x does below $\ell(x) + 4$, which proves $\Delta(x+20) = \Delta(x)$ for all $x > 103$. ■

With a slightly more refined argument than the one given above, one can prove without the use of software that $\Delta(x+20) = \Delta(x)$ for all $x \geq 92$. We encourage the interested reader to work out such an argument.

Calculating ω-primality. We conclude our study of 🐔 with an expression for the ω-primality of $x \in$ 🐔 and show (in some sense) how far a McNugget number is from being prime. We proceed in a similar fashion to Theorems 6 and 7, showing that with only two exceptions, $\omega(x + n_1) = \omega(x) + 1$ for all $x \in$ 🐔.

Theorem 9. *With the exception of* $x = 6$ *and* $x = 12$, *every nonzero* $x \in$ 🐔 *satisfies* $\omega(x+6) = \omega(x) + 1$. *In particular, we have*

$$\omega(x) = \begin{cases} q & \text{if } r = 0, \\ q+5 & \text{if } r = 1, \\ q+7 & \text{if } r = 2, \\ q+2 & \text{if } r = 3, \\ q+4 & \text{if } r = 4, \\ q+9 & \text{if } r = 5, \end{cases}$$

for each $x \neq 6, 12$, *where* $x = 6q + r$ *for* $q, r \in \mathbb{N}$ *and* $r < 6$.

Proof. Fix $x \in$ 🐔. Following the spirit of the proof of Theorem 6, we begin by proving each $x > 12$ has a maximum length bullet (a, b, c) with $a > 0$. Indeed, suppose $(0, b, c)$ is a bullet for x for some $b, c \geq 0$. The element $x \in$ 🐔 also has some bullet of the form $(a', 0, 0)$, where a' the smallest integer such that $6a' - x \in$ 🐔. Notice $a' \geq 3$ since $x > 12$. We consider several cases.

- If $c = 0$, then $9b - x \in$ 🐔 but $9b - x - 9 \notin$ 🐔. If $b \leq 3$, then $a' \geq b$. Otherwise, either $9(b-1)$ or $9(b-2)$ is a multiple of 6, and since $9(b-2) - x \notin$ 🐔 as well, we see $a' \geq \frac{3}{2}(b-2) + 1 \geq b$.

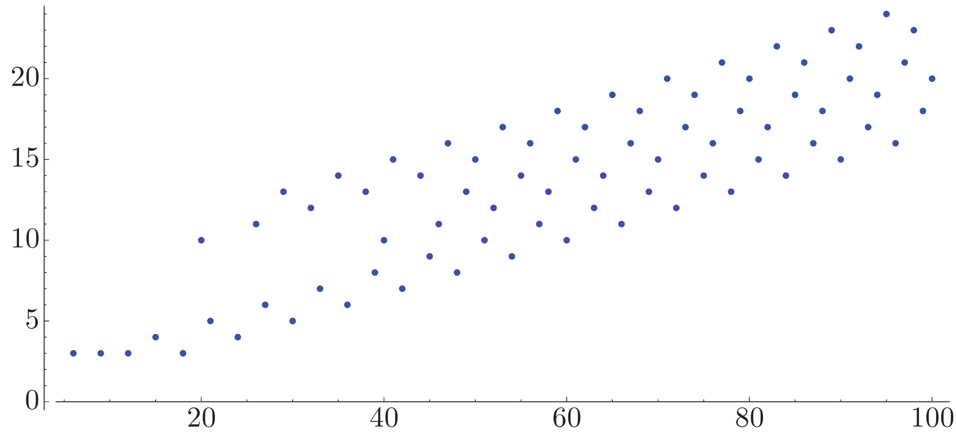

Figure 2 A plot depicting the ω-primality function $\omega(n)$ for $n \in$ 🐔.

- If $b = 0$, then there are two possibilities. If $c \leq 3$, then $a' \geq c$. Otherwise, one of $20(c-1)$, $20(c-2)$ and $20(c-3)$ is a multiple of 6, so we must have that $a' \geq \frac{10}{3}(c-3) + 1 \geq c$.
- If $b, c > 0$, then $9b + 20c - x - 9, 9b + 20c - x - 20 \notin$ 🐔, so $9b + 20c - x$ is either 0, 6, or 12. This means either $(3, b-1, c)$, $(2, b-1, c)$, or $(1, b-1, c)$ is also a bullet for x, respectively.

In each case, we have a bullet for x at least as long as $(0, b, c)$, but with positive first coordinate, so we conclude x has a maximal bullet with nonzero first coordinate.

Using a similar argument to that given in the proof of Theorem 6, if $(a+1, b, c)$ is a maximum length bullet for $x + 6$, then (a, b, c) is a maximum length bullet for x. This implies $\omega(x+6) = \omega(x) + 1$ whenever $x + 6$ has a maximum length bullet with positive first coordinate, which by the above argument holds whenever $x > 12$. This proves the first claim.

The formula for $\omega(x)$ now follows from the first claim, computations $\omega(9) = 3$ and $\omega(20) = 10$ given above, and analogous computations for $\omega(15) = 4$, $\omega(18) = 3$, $\omega(29) = 13$, $\omega(40) = 10$, and $\omega(49) = 13$. ∎

Figure 2 plots McNugget monoid element ω-values. Since $\omega(x+6) = \omega(x) + 1$ for large $x \in$ 🐔, most of the plotted points occur on one of 6 lines with slope $\frac{1}{6}$. It is also evident in the plot that $x = 6$ and $x = 12$ are the only exceptions.

Although $\omega(x+6) = \omega(x) + 1$ does not hold for every $x \in$ 🐔, there are some numerical monoids for which the "sufficiently large" hypothesis in Theorem 5 can be dropped (for instance, any numerical monoids with 2 minimal generators has this property). Hence, we conclude with a problem suitable for attack by undergraduates.

Question. Determine which numerical monoids $\langle n_1, \ldots, n_k \rangle$ satisfy each of the following conditions for all x (i.e., not just sufficiently large x):

1. $L(x + n_1) = L(x) + 1$,
2. $\ell(x + n_k) = \ell(x) + 1$, or
3. $\omega(x + n_1) = \omega(x) + 1$.

Appendix: computer software for numerical monoids

Many of the computations referenced in this article can be performed using the `numericalsgps` package [17] for the computer algebra system `GAP`. The brief snippet of sample code below demonstrates how the package is used to compute various quantities discussed in this paper.

```
gap> LoadPackage("num");
true
gap> McN:=NumericalSemigroup(6,9,20);
<Numerical semigroup with 3 generators>
gap> FrobeniusNumberOfNumericalSemigroup(McN);
43
gap> 43 in McN;
false
gap> 44 in McN;
true
gap> FactorizationsElementWRTNumericalSemigroup(18,McN);
[ [ 3, 0, 0 ], [ 0, 2, 0 ] ]
gap> OmegaPrimalityOfElementInNumericalSemigroup(6,McN);
3
```

This only scratches the surface of the extensive functionality offered by the `numericalsgps` package. We encourage the interested reader to install and experiment with the package; instructions can be found on the official `GAP` webpage:

https://www.gap-system.org/Packages/numericalsgps.html

Acknowledgments This article is based on a 2013 PURE Mathematics REU Project by Emelie Curl, Staci Gleen, and Katrina Quinata which was directed by the authors and Roberto Pelayo.

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Summary. Every day, 34 million Chicken McNuggets are sold worldwide. At most McDonalds locations in the United States today, Chicken McNuggets are sold in packs of 4, 6, 10, 20, 40, and 50 pieces. However, shortly after their introduction in 1979, they were sold in packs of 6, 9, and 20. The use of these latter three numbers spawned the so-called Chicken McNugget problem, which asks: "What numbers of Chicken McNuggets can be ordered using only packs with 6, 9, or 20 pieces?" In this article, we present an accessible introduction to this problem, as well as several related questions whose motivation comes from the theory of nonunique factorization.

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An Alternate Approach to Changing Sine

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A review of standard calculus textbooks, e.g., [2], [10], and [11], reveals a common approach to showing that the derivative of the sine function is the cosine function. This approach is to derive $\lim_{x\to 0}(\sin x)/x = 1$, obtain the corollary $\lim_{x\to 0}(\cos x - 1)/x = 0$, and then use both in the limit definition of derivative after applying a trigonometric identity for the sine of a sum. While there is some variation in the texts as to how the first limit is obtained using geometry, the final steps using a trigonometric identity and applying both limits is common.

The goal here is to advocate for an alternative approach that reduces the use of trigonometric identities and provides good motivation for this derivative as a rate of change. This idea is not new, as others (e.g., in [4], [5], [6], and [7]) begin with a similar approach and with a figure similar to our Figure 1. Indeed, Hesterberg [5] reports that the principle is found an 1870s textbook [9]. We provide slightly different geometric arguments for the inequalities that are similar to some of the standard approaches to evaluating $\lim_{h\to 0}(\sin h)/h = 1$, which directly shows that the derivative of sine at 0 is 1 (equal to $\cos 0$), shifted along the unit circle. Other unit circle approaches to this derivative and more appear in [3] and [8], while Boyer [1] provides a history.

Rate of change of height along the unit circle

Recalling that the derivative is a rate of change and that values of the sine function are "heights" (y coordinates) on the unit circle we see that the derivative of sine tells us "how fast" the "height" changes as we "walk" at unit speed along the unit circle.

First, we look at the translation to the unit circle of a secant line on the graph of $y = \sin x$. Using common calculus notation, we consider that slope is "rise/run" for "inputs" x and $x + h$, where for now we assume h is positive and x is between 0 and $\pi/2$.

The slope of the secant on the graph $y = \sin x$ is $(\sin(x + h) - \sin x)/h$. Translated to the unit circle, this is the ratio of the change in height to the length of the arc. That is, $|AC|/h$ in Figure 1. Here, we get our first hint that the derivative should be cosine. If we believe that as h gets small the arc AD in the region ACD approaches a straight line so the region approaches a triangle with, as will become more clear in Figure 2, the angle between the vertical segment and hypotenuse $x + h$ (which approaches x), then $|AC|/h \to \cos(x + h) \to \cos x$.

Of course, this intuition does not provide a proof. For this we bound the slope of the secant using two simple geometric arguments with slightly different triangles.

Secant limit for positive change h in the first quadrant

On the left in Figure 2, we zoom to the first quadrant region along our unit circle between the heights $\sin x$ and $\sin(x + h)$. The line AB is the tangent line to the circle

Math. Mag. **91** (2018) 337–340. doi:10.1080/0025570X.2018.1515558
MSC: Primary 26A06, Secondary 97I40

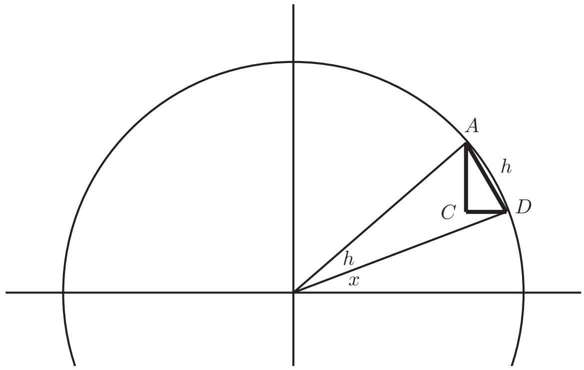

Figure 1 Difference quotient for sin(x) viewed on the unit circle.

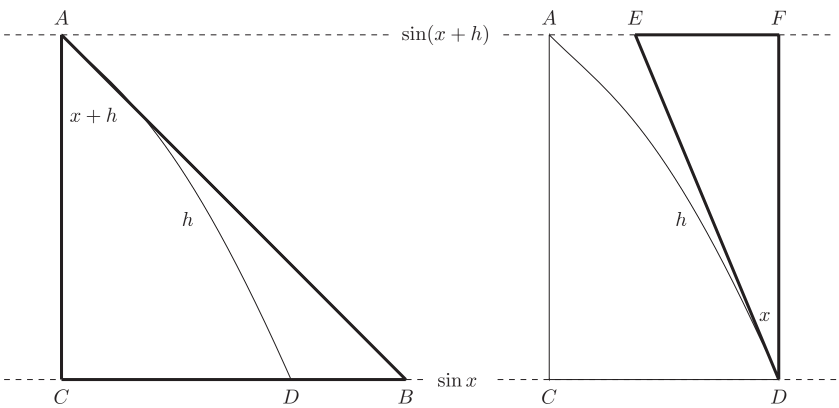

Figure 2 Bounding the difference quotient.

at A. Accept for now that length h of arc AD is less than the length of the segment AB. We will establish this later. Also observe from basic trigonometry that the angle $\angle BAC$ has measure $x + h$. Then using $h < |AB|$

$$\frac{\sin(x+h)-\sin x}{h} > \frac{\sin(x+h)-\sin x}{|AB|} = \frac{|AC|}{|AB|} = \cos(x+h).$$

In a similar manner, using the right half of Figure 2, we get

$$\frac{\sin(x+h)-\sin x}{h} < \frac{\sin(x+h)-\sin x}{|DE|} = \frac{|DF|}{|DE|} = \cos x.$$

Now, for $0 \leq x < \frac{\pi}{2}$ and small positive h, we have that

$$\cos(x+h) < \frac{\sin(x+h)-\sin x}{h} < \cos x.$$

By the squeeze theorem, we get

$$\lim_{h\to 0^+} \frac{\sin(x+h)-\sin x}{h} = \cos x.$$

Arc inequalities

We establish the arc bounds as follows.

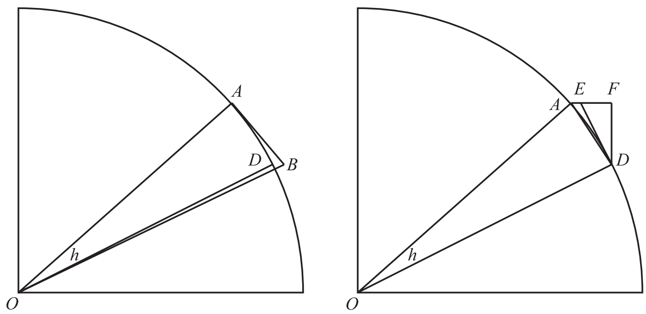

Figure 3 Bounding arc length h.

On the left in Figure 3, the area of sector OAD is $\frac{h}{2\pi} \cdot \pi \cdot 1^2 = \frac{h}{2}$ and the area of triangle OAB is $\frac{1}{2} \cdot 1 \cdot |AB| = \frac{|AB|}{2}$. Since right triangle OAB contains sector OAD,

$$\frac{h}{2} = \text{ area of sector } OAD < \text{ area of triangle } OAB = \frac{|AB|}{2}$$

establishing the first arc inequality $h < |AB|$.

On the right in Figure 3, as angle $\angle ADF$ is greater than angle $\angle EDF$, using cosines $\frac{|DF|}{|DA|} < \frac{|DF|}{|DE|}$. Hence $|DA| > |DE|$. Then, since the arc AD length h is greater than the segment length $|DA|$, we get $h > |DA| > |DE|$ establishing the second arc inequality $h > |DE|$.

Left limit and all quadrants

To complete a proof, we would need to consider the left side limit and x in other quadrants.

For the left limit consider $h < 0$. The figures are nearly the same as those above for $h > 0$ and are omitted. For the new versions, the top height is $\sin x$, the angle $\angle CAB$ is x, the bottom height is $\sin(x+h)$ and angle $\angle FDE$ is $x+h$. In addition, the arc length is now $-h$ as $h < 0$. The secant slope is $(\sin x - \sin(x+h))/(-h) = (\sin(x+h) - \sin x)/h$. From the switching of the angles as described above, we end up with $\cos x < (\sin(x+h) - \sin x)/h < \cos(x+h)$ when $h < 0$. Then, from the squeeze theorem, we get $\lim_{h\to 0^-}(\sin(x+h) - \sin x)/h = \cos x$ as needed.

To extend to angles beyond the first quadrant on the unit circle, we can easily draw appropriate figures symmetric to those above, make appropriate changes in values and get the limit as $\cos x$. We omit the straightforward details for this.

An alternative informal explanation for other quadrants (not proof) is that we have shown that while moving along the unit circle at unit speed in the first quadrant, the rate of increase of height is the horizontal distance from the vertical axis. In the fourth quadrant, we get the same rate of increase and the same horizontal distance $\cos x$. In the second and third quadrants, the "speed" of the rates are the same but the height is decreasing as the distance along the circle increases. Hence, the rate of change is the negative of the horizontal distance, which is $\cos x$.

Acknowledgments The author thanks the reviewers for providing the references as well as many useful suggestions including the wording for the first paragraph on the secant limit.

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Summary. We present an alternate approach to the derivative of the sine function using unit circle geometry to conceptualize why the derivative of sine is cosine.

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Pietro Mengoli's 1650 Proof that the Harmonic Series Diverges

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The first published proof that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

exceeds any given quantity was given by Pietro Mengoli in 1650 [9]. The same result had been proved by Nicole Oresme in Question 2 of his *Questiones super geometriam Euclidis* [7, pp. 131–135], dated around 1350. These *Questiones* were copied as a manuscript but were not published until the 20th century. There is no indication that Mengoli knew of this work. Oresme's proof is the one still commonplace today, based on grouping the terms of the series into blocks of 2 terms, 4 terms, 8 terms, etc. Mengoli's proof is also based on the grouping of terms, but in a different manner. He groups the terms into blocks of three and applies the inequality

$$\frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} > \frac{3}{n}. \tag{1}$$

This inequality follows, as Mengoli says, from the fact that the first term exceeds the middle by more than the middle exceeds the last, i.e., $\frac{1}{n-1} - \frac{1}{n} > \frac{1}{n} - \frac{1}{n+1}$, and therefore, replacing the outer terms by the middle one will diminish the first by more than it will increase the last.

Applying this inequality to the harmonic series gives

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots &= 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \cdots \\ &> 1 + \frac{3}{3} + \frac{3}{6} + \cdots \\ &= 1 + 1 + \frac{1}{2} + \cdots \end{aligned}$$

Note that the harmonic series recurs in the final expression: if we let S denote the sum of the harmonic series, we have just proved that $S > 1 + S$. From this inequality, it follows that S cannot be finite, so this is one way of arriving at the desired result. Indeed, most modern accounts of Mengoli's proof put this exact reasoning in his mouth. In particular, [4, pp. 7–10], [8], and [1, pp. 11–12] all phrase Mengoli's proof in this exact way.

We believe it is highly unfortunate that this has become the standard account of Mengoli's proof, for in fact he does *not* argue in this way, and indeed the fact that

Math. Mag. **91** (2018) 341–347. doi:10.1080/0025570X.2018.1506656

he does not do so is arguably one of the most interesting and historically illuminating aspects of his proof. Below we give a complete English translation of Mengoli's argument, so that it may be appreciated in his own terms and its persistent misrepresentations eradicated.

In the fourth paragraph of the translation, we see that Mengoli does indeed note and utilise the self-replicating nature of the above estimation procedure. However, he does not treat as an entity the completed series in the manner of the inequality $S > 1 + S$. Instead he notes that (1) tells us that the sum of the first three terms,

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \tag{2}$$

is greater than $\frac{3}{3}$, i.e., greater than 1, and the sum of the next nine terms,

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13}, \tag{3}$$

is greater than

$$\frac{3}{6} + \frac{3}{9} + \frac{3}{12},$$

i.e., greater than the sum (2), and the sum of the next 27 terms is greater than

$$\frac{3}{15} + \frac{3}{18} + \frac{3}{21} + \frac{3}{24} + \frac{3}{27} + \frac{3}{30} + \frac{3}{33} + \frac{3}{36} + \frac{3}{39},$$

i.e., greater than the sum (3), etc. Therefore, when the terms of the original series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$$

are grouped into blocks of size 3, 9, 27, etc., the sum of each block exceeds 1, since by repeated application of (1) the sum of each such block is greater than the sum of the previous block and thus greater than the sum of the first three terms and hence greater than 1. (Note that Mengoli does not include a leading term 1, as is customary today when talking about the harmonic series.)

Thus a faithful schematic representation of Mengoli's proof is not to conclude from $S > 1 + S$ that S cannot be finite, but rather to apply this inequality repeatedly to yield $S > 1 + S > 2 + S > 3 + S > \cdots$, and a fortiori $S > 1$, $S > 2$, $S > 3$, etc., from which it follows that the series continued sufficiently far can be made to exceed any given quantity. The accounts of Mengoli's proof given by [**11**], [**5**], [**10**, pp. 14–23], and [**2**, pp. 204–205] capture this aspect of the proof much more faithfully than the sources cited above.

But even the very idea of considering the sum of the series as a number or algebraic entity that can be denoted by a single symbol such as S is foreign to Mengoli. And that with good reason. For what grounds do we have for assuming that an infinite series can be considered as a unified algebraic entity and be operated on as such? Mengoli's approach dextrously avoids all the potential pitfalls of dealing with infinities in a careless fashion. Instead of speaking in abstractions such as saying that the harmonic series equals infinity, he remains thoroughly finitistic, saying that the series exceeds 1 if you take 3 terms, 2 if you take $3 + 9$ terms, 3 if you take $3 + 9 + 27$ terms, etc. This captures the infinity of the series in the most concrete and constructive manner possible, in unequivocal terms that are not susceptible to any philosophical qualms about infinities.

Mengoli's cautious approach to the infinite is very much in keeping with the way the infinite was treated in classical Greek mathematics. In particular, the Greek "method of exhaustion" was a fundamental technique for avoiding appeal to the infinite. Greek mathematicians used this technique to determine many areas by, in effect, limiting processes of polygonal approximations. But to avoid explicit use of the infinite and instead phrase their results in safe, finitistic terms, they showed, by a double reduction ad absurdum, that all other possible values for the area, except the one claimed in the theorem at hand, would be impossible. To rule out any given value for the area other than the correct one, only a finite number of steps in the polygonal approximation would be needed, whence the proof avoids assuming the completion of an infinite number of operations or making any explicit reliance on the infinite. In other words, the method of exhaustion deals only with the *potential* infinity (as Aristotle called it) of the procedure having the potential of being extended indefinitely, as opposed to the *actual* infinity of considering the approximation process or series as having been carried through to its completion.

Indeed, Mengoli cites Archimedes' *Quadrature of the Parabola* as having occasioned his work on infinite series. This is a prime example of the method of exhaustion. In it, Archimedes shows that the area of a segment of a parabola can be approximated by inscribed triangles in such a way that the triangles added at each step of the approximation have one quarter the area of those at the preceding step. Thus the total area of the parabolic segment is

$$A + \frac{1}{4}A + \frac{1}{4^2}A + \frac{1}{4^3}A + \cdots = \frac{4}{3}A,$$

where A is the area of the initial inscribed triangle (which itself is straightforward to determine). Again, Archimedes does not speak of the sum of an infinite series but rather shows that, by bringing the approximation far enough, any other possible value for the area is ruled out.

Mengoli was surely very sensitive to this context. In fact, as Eneström [3] points out, this even explains the title of Mengoli's work, which is called "New arithmetical quadratures" even though it contains no actual quadratures (i.e., area determinations). Thus Mengoli evidently associated the theory of series very closely with the classical method of exhaustion, so it is not surprising that he remains committed to its finitistic paradigm.

The nature of Mengoli's proof makes it a perfect showcase for the great importance attached to these considerations at the time. For if there ever were a time to employ a form of reasoning (such as that $S > 1 + S$ implies $S = \infty$) that considers a series as a single, completed algebraic entity, then this was it. Indeed, as we have pointed out, the temptation to reason in this way in this case is so strong that even several writers on the history of mathematics have succumbed to it when describing Mengoli's proof. Thus the fact that Mengoli does *not* do so is an especially telling testament to his dedication to the ancient manner of dealing with infinities in strictly finitistic terms. It is a great pity, therefore, that this crucial aspect of his proof is misrepresented in the standard modern accounts of it.

The translation

Mengoli's proof of the divergence of the harmonic series occurs in the first five paragraphs of the Preface of his 1650 *Novae quadraturae arithmeticae* [9]. We now give a complete English translation of this passage. The translation will be followed by some explanatory notes.

Meditating often on Archimedes' quadrature of the parabola, in which infinitely many triangles, being in continued quadruple proportion, do not exceed certain bounds, the universal quadrature came to mind, demonstrated by geometers using the same proof, in which infinitely many magnitudes in some continued proportion of greater inequality are gathered into determined homogeneous quantities. This admirable theorem! In contemplating it, I was led to the question, whether magnitudes arranged under such a rule, whatever it may be, such that some can be taken smaller than any given quantity, or that decreasing terms vanish *in infinitum*, when composed infinitely can exceed each given quantity.

Having gone about to try arithmetical fractions for the purpose of such an experiment, I set them out thus, so that all the unities are denominated by all the numbers after unity,

$$\frac{1}{2}\ \frac{1}{3}\ \frac{1}{4}\ \frac{1}{5}\ \frac{1}{6}\ \frac{1}{7}\ \frac{1}{8}\ \frac{1}{9}\ \frac{1}{10}\ \frac{1}{11}\ \frac{1}{12}\ \frac{1}{13}\ \frac{1}{14}.$$

In this arrangement, the magnitude can be taken less than any given amount, and therefore these magnitudes decreasing in quantity according to the increase of the rank, disappear into infinity.

Propounding the question in the terms of the assumed arrangement, I was therefore searching for an argument to decide whether the unities denominated by every number starting with unity, laid out to infinity, taken together would make up some infinite or finite extent. It seemed that the answer would have to be in favor of a finite extent, since the powers of numbers and of fractions are opposed: that of numbers in multiplication, by which quantities progress towards infinity, but that of fractions in division, by which a thing is reduced downright to indivisibles: now the numbers taken together exceed any given quantity; so by the opposite reasoning it seems that the fractions cannot exceed any given quantity. This sophism was the reason of my expectation, held for almost an entire month, that I would be able to decide in favor of this geometrical view about the matter; but when I now examine the procedure of proof, my judgment changes to the other view.

The procedure is the following. Because in the given fractions equal magnitudes are denominated by numbers in arithmetic proportion, and thus three consecutive terms, say A, B, and C, are in harmonic proportion, for example,

$$\begin{matrix} A & B & C \\ \tfrac{1}{2} & \tfrac{1}{3} & \tfrac{1}{4} \end{matrix}$$

and A has the same proportion to C as the excess of A to B has to the excess of B to C, and moreover, A is greater than C, therefore the excess of A to B is greater than the excess of B to C. The total of A and C is greater than twice B, and the total of the three A, B, C is greater than thrice the middle term B. So by this argument, the fractions in this arrangement taken three at a time from the first,

$$\frac{1}{2}\ \frac{1}{3}\ \frac{1}{4} \qquad \frac{1}{5}\ \frac{1}{6}\ \frac{1}{7} \qquad \frac{1}{8}\ \frac{1}{9}\ \frac{1}{10} \qquad \frac{1}{11}\ \frac{1}{12}\ \frac{1}{13} \qquad \frac{1}{14}\ \frac{1}{15}\ \frac{1}{16},$$

are greater than thrice the middle terms: and the middle terms are unities denominated by numbers multiplied by three, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{9}$, $\frac{1}{12}$, and thrice these are 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, the same ones which in the above argument taken three at a time are greater than thrice the middle terms. Therefore the given fractions of the arrangement, taken according to numbers in subtriple proportion, 3, 9, 27, 81, all exceed unities. For any given number, one can take equally many numbers in continued subtriple proportion starting from three, and then the fractions of the proposed arrangement taken according to the sum of the numbers in continued proportion will exceed the given number. Therefore the proposed fractions, arranged up to infinity and taken together, are capable of filling an infinite extent.

For example, let 4 be the assigned number, and starting from three take four numbers in continued subtriple proportion, 3, 9, 27, 81, whose sum is 120: then 120 of the given fractions exceed the assigned number 4. For, the first three exceed thrice $\frac{1}{3}$, namely unity; the next nine exceed thrice the sum of $\frac{1}{6}, \frac{1}{9}, \frac{1}{12}$, namely the sum of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, but as I have shown the sum of those exceeds unity, and so these nine exceed unity; and by the same demonstration the following 27 and 81 exceed unities.

Notes on the text

Mengoli opens his discussion with a reference to Archimedes' *Quadrature of the Parabola*. This shows the context in which he was led to the study of series, namely using them—in particular geometric series—for the determination of areas according to the method of exhaustion. This explains why he uses the term "quadrature" to mean, in effect, the sum of a series, even though this term normally means finding an area. Indeed, as noted above, the "new quadratures" promised in the title of Mengoli's work are quadratures only in the sense of summing series. Mengoli's interest is not in the geometrical application of series. In the above translated paragraphs, he instead poses a more abstract question which arises from reflecting on such series, namely the question of whether the sum of an infinite series can exceed any quantity even though its terms become smaller and smaller and approach zero.

In the terminology of the first paragraph, then, "universal quadrature" does not mean finding the area of any figure, but rather finding the sum of any sequence of magnitudes in a "continued proportion of greater inequality," this being "universal" in that it generalizes the sequence of areas Archimedes used in which each was one-quarter the previous. To say that homogeneous magnitudes $a_1, a_2, a_3, \ldots$ are in *continued proportion* means that $a_1 : a_2 = a_2 : a_3$, $a_2 : a_3 = a_3 : a_4$, etc. In other words, a sequence of homogeneous magnitudes $a_1, a_2, a_3, \ldots$ is in continued proportion when there is some dimensionless quantity r such that $a_1 = ra_2$, $a_2 = ra_3$, $a_3 = ra_4$, etc. "Quadruple proportion" means $r = 4$. To say that a ratio $a : b$ is in *greater inequality* means that $a > b$. Thus, the "universal quadrature" means summing any geometric series with strictly decreasing terms. To speak about the magnitudes "composed infinitely" means summing the magnitudes. Heath [6, p. 85] may be consulted for a summary of the notions of arithmetic, geometric, and harmonic proportions in classical Greek mathematics.

In the second paragraph, by "arithmetical fractions" Mengoli means that the denominators are in *arithmetic proportion* rather than geometric proportion, i.e., the terms are of the form $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}, \ldots$ where $a_1, a_2, a_3, a_4, \ldots$ satisfy $a_1 - a_2 = a_2 - a_3$, $a_2 - a_3 = a_3 - a_4$, etc. "Decreasing in quantity according to the increase of the rank" means that the greater the index of a term, the smaller the magnitude of the term.

In the third paragraph, Mengoli states that he at first believed that the harmonic series must have a finite value since when one makes a number bigger and bigger, the corresponding fraction becomes smaller and smaller, and thus eventually "indivisible", so that it would contribute nothing to a sum. Mengoli's point that he was misled by this "sophism" (i.e., a confusing or deceptive argument) "for almost an entire month" serves to make the reader appreciate the counterintuitive nature of his result.

The fourth paragraph contains the proof that the harmonic series diverges. In symbols, Mengoli says that for A, B, and C to be in harmonic proportion means that

$$\frac{A}{C} = \frac{A-B}{B-C}.$$

If a, b, and c are in arithmetic progression, i.e., $a - b = c - b$, then their reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in harmonic proportion:

$$\frac{\frac{1}{a} - \frac{1}{b}}{\frac{1}{b} - \frac{1}{c}} = \frac{\frac{b-a}{ab}}{\frac{c-b}{bc}} = \frac{b-a}{c-b}\frac{c}{a} = \frac{c}{a} = \frac{\frac{1}{a}}{\frac{1}{c}}.$$

If A, B, and C are in harmonic proportion and $A > C$, then $A - B > B - C$, so $A + C > 2B$ and hence $A + B + C > 3B$. Using this fact that applies to triples in harmonic proportion, we chunk the sequence $\frac{1}{2}, \frac{1}{3}, \ldots$ into triples of the form $\frac{1}{3n-1}, \frac{1}{3n}, \frac{1}{3n+1}$, and the sum of each triple is greater than thrice the middle term, i.e., greater than $\frac{1}{n}$.

Thus, first, the sum of the three terms

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$$

is greater than $\frac{3}{3} = 1$. Second, the sum of $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}$ is greater than $\frac{3}{6} = \frac{1}{2}$, the sum of $\frac{1}{8}, \frac{1}{9}, \frac{1}{10}$ is greater than $\frac{3}{9} = \frac{1}{3}$, and the sum of $\frac{1}{11}, \frac{1}{12}, \frac{1}{13}$ is greater than $\frac{3}{12} = \frac{1}{4}$, so the sum of the nine terms

$$\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13} \tag{4}$$

is greater than the sum of

$$\frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}.$$

But we know that the latter sum is itself greater than 1, so the sum of (4) is greater than 1. Third, the sum of $\frac{1}{14}, \frac{1}{15}, \frac{1}{16}$ is greater than $\frac{3}{15} = \frac{1}{5}$, the sum of $\frac{1}{17}, \frac{1}{18}, \frac{1}{19}$ is greater than $\frac{3}{18} = \frac{1}{6}$, etc., and the sum of $\frac{1}{38}, \frac{1}{39}, \frac{1}{40}$ is greater than $\frac{3}{39} = \frac{1}{13}$, so the sum of the 27 terms

$$\frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \frac{1}{18}, \frac{1}{19}, \ldots, \frac{1}{38}, \frac{1}{39}, \frac{1}{40} \tag{5}$$

is greater than the sum of

$$\frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{13}.$$

But we know that the latter sum is itself greater than 1, so the sum of (5) is greater than 1. Thus, Mengoli chunks the sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \tag{6}$$

into blocks with 3, 9, 27, etc. terms, and the sum of the terms in each block is greater than 1. There are infinitely many blocks, and therefore the sum of (6) is greater than $1 + 1 + 1 + \cdots$, namely it fills an "infinite extent."

The ratio 1 : 3 is a "subtriple proportion," and saying that the numbers 3, 9, 27, 81 are in subtriple proportion means that the consecutive ratios 3 : 9, 9 : 27, 27 : 81 are subtriple proportions.

In the fifth paragraph Mengoli spells out an explicit recipe for how many terms will suffice for the series to exceed any given number. Generally, for a positive integer n, the sum of the first $\sum_{k=1}^{n} 3^k = \frac{3^{n+1}-3}{2}$ terms of the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is greater than n. Mengoli gives as an example $n = 4$, for which $\frac{3^{4+1}-3}{2} = 120$, and $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{121} > 4$. (In fact, ones computes that this sum is equal to $5.368\ldots$.) Again, this shows very clearly Mengoli's commitment to a finitistic or constructive notion of what it means for a sum to be infinite.

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Summary. Pietro Mengoli proved that harmonic series diverges in his 1650 *Novae quadraturae arithmeticae* – the first published proof of this result. His proof is discussed in a number of places in the secondary literature, but is often misrepresented in a crucial manner. We give a full English translation of the proof with explanatory notes and argue for a less anachronistic interpretation of it.

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How Do You Fix an Oval Track Puzzle?

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The Top Spin puzzle was invented by Binary Arts (now Think Fun) in 1989. The game consists of an oval track containing 20 tiles, numbered 1 through 20, and an intersecting turntable which holds 4 tiles at a time (see Figure 1).

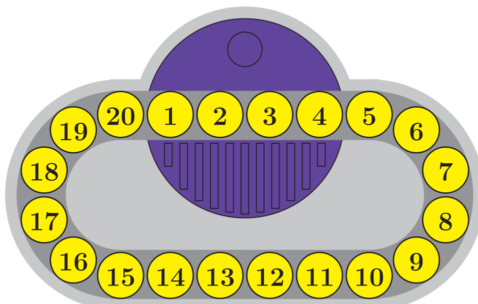


Figure 1 The (solved) Top Spin puzzle.

You can push the tiles around the track and the turntable allows you to flip four tiles at a time. Starting with the numbers mixed up, your goal is to put them back in increasing order as you move clockwise around the track. If we fix the left side of the turntable as the first position, then puzzle arrangements naturally correspond to permutations of the numbers $\{1, 2, \ldots, 20\}$. Thus, we can think of puzzle arrangements as sitting inside of the symmetric group S_{20}.

Of course, we can generalize the puzzle as well by considering the set of tiles $\{1, \ldots, n\}$ for any natural number $n \in \mathbb{N}$ and by allowing for a turntable of size k for any natural number $1 \leq k \leq n$. These more general puzzles are often referred to as oval track puzzles and, just as in the Top Spin case, we can view puzzle arrangements as elements of the symmetric group S_n.

Now, imagine that you own one of these oval track puzzles and that you loan it out to a friend one day. While attempting to solve the puzzle your friend becomes frustrated and smashes the puzzle on the table, thereby knocking out all of the tiles. For a brief moment you consider yelling (to get your own frustration out), but your innate curiosity gets the better of you. Your friend has inadvertently brought a fascinating conundrum to your attention. You'd like to put your puzzle back together, but you want it to still be solvable (otherwise it will forever be a source of *only* frustration). Does it matter how you put the tiles back in?

Math. Mag. **91** (2018) 348–358. doi:10.1080/0025570X.2018.1506657
MSC: Primary 20B35, Secondary 00A08; 05A05
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/umma.

As it turns out, if the puzzle you own is of the standard Top Spin variety, then the answer is no. No matter how you put the tiles back in, it will always be solvable. However, for some of the more general puzzles, it does depend on how you replace the tiles. In the rest of the article, we answer this question in general by describing how the tiles must be replaced in order to "fix" a broken puzzle in each case. Solving the puzzle corresponds to obtaining the identity permutation in S_n. Thus, we determine which puzzle arrangements are solvable by considering the two moves in the oval track puzzle—the *translation* of the n tiles around the track and the *flip* of k tiles using the turntable—as generators of a subgroup of S_n. For $n, k \in \mathbb{N}$ with $1 \leq k \leq n$, we will refer to the subgroup obtained as the *oval track group*, $OT_{n,k}$.

Preliminaries

We start with a couple of important results known about the symmetric group which we plan to use, but we highly encourage any reader who is unfamiliar with the symmetric group to learn more about this beautiful group (see nearly any undergraduate text on abstract algebra, e.g., [1] or [5]). The first result is a well-known (see, e.g., [6, Lemma 2.7]) and extremely useful fact which describes what happens when you conjugate by permutations in the symmetric group.

Lemma 1. Given $\theta, \sigma \in S_n$, then $\theta\sigma\theta^{-1}$ is the permutation obtained from σ by applying θ to every number that appears in the disjoint cycle representation of σ.

We will make use of conjugation several times throughout the article so we give a quick example of Lemma 1 in action. Let $\theta = (1\ 2\ 3\ 4)$ and $\sigma = (1\ 3\ 5)(2\ 6)$ be permutations in S_6, then

$$\theta\sigma\theta^{-1} = (\theta(1)\ \theta(3)\ \theta(5))(\theta(2)\ \theta(6)) = (2\ 4\ 5)(3\ 6).$$

Since we aim to describe the group that our two basic moves generate, it will be helpful to know quickly when we generate the alternating group in various contexts. More specifically, we will use the fact that (for $n \geq 3$) the elements of A_n can be generated by a single consecutive 3-cycle together with the n-cycle $(1\ 2\ \ldots\ n)$.

Proposition 1. If $n \geq 3$ and $(i\ i+1\ i+2)$ is any consecutive 3-cycle, then A_n is a subgroup of the group generated by $(i\ i+1\ i+2)$ and the n-cycle $(1\ 2\ \ldots\ n)$.

Proof. It is well known (see, e.g., [2, Lemma 2]) that the set of consecutive 3-cycles will generate all of A_n. The result, therefore, follows as we may generate all of the consecutive 3-cycles using conjugation by the n-cycle. ■

The parity subgroup. Observe that for any $n \in \mathbb{N}$, we can consider permutations in S_n which maintain parity, i.e., ones which permute the odd numbers amongst themselves and simultaneously permute the even numbers amongst themselves. We'll refer to these as *Type I* permutations. For example, our permutation $\sigma = (1\ 3\ 5)(2\ 6)$ from earlier is Type I. If we isolate the odd portion of a Type I element, then we may think of it as a permutation in the symmetric group on the first $\lceil \frac{n}{2} \rceil$ odd numbers, which we denote by $S^{\text{odd}}_{\lceil \frac{n}{2} \rceil}$. Similarly, the even portion will be a permutation of the first $\lfloor \frac{n}{2} \rfloor$ even numbers, $S^{\text{even}}_{\lfloor \frac{n}{2} \rfloor}$. Thus, the Type I elements in S_n exactly correspond to the elements in $S^{\text{odd}}_{\lceil \frac{n}{2} \rceil} \times S^{\text{even}}_{\lfloor \frac{n}{2} \rfloor}$ and it follows that there are $\left(\lceil \frac{n}{2} \rceil\right)!\left(\lfloor \frac{n}{2} \rfloor\right)!$ of these permutations.

In the special case when n is even, it is also possible to have permutations which exactly swap the parity, i.e., ones which send all odd numbers to even numbers and vice versa. We'll refer to these as *Type II* permutations. For example, the permutation $\theta = (1\ 2\ 3\ 4)$ is a Type II permutation in S_4, however, it is **not** Type II in S_6 since in that group θ fixes 5 and 6 rather than changing their parity. Observe that there are exactly $\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!$ different Type II permutations in S_n as well.

When taken together, the union of Type I and Type II permutations, denoted by PS_n, forms what we call the *parity subgroup* of S_n. Thus, for any even $n \in \mathbb{N}$, we have $PS_n = \{\theta \in S_n \mid \theta \text{ is Type I or Type II}\}$. Our observations above imply that $|PS_n| = 2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!$.

Getting familiar with the oval track group. While we will make use of tools and notation from the symmetric group, it will also be helpful to think about the oval track group from the perspective of puzzle arrangements. From that point of view, we think of applying group elements as moves on the puzzle. Towards that end, we will represent arrangements as diagrams. For example, the solved arrangement (or $(1) \in S_n$) is

$$\left(\textcircled{1}\cdots\textcircled{k}\right)\textcircled{k+1}\cdots\textcircled{n},$$

where the parentheses mark the location of the turntable.

Recall that the two basic moves that we have at our disposal are the *translation*, which we will denote by τ, and the *flip*, which we will denote by ϕ. The translation rotates all tiles one position clockwise around the track, hence diagramatically:

$$\tau\left(\textcircled{1}\cdots\textcircled{k}\right)\textcircled{k+1}\cdots\textcircled{n} = \left(\textcircled{n}\,\textcircled{1}\cdots\textcircled{k-1}\right)\textcircled{k}\cdots\textcircled{n-1}.$$

Observe that n has moved to the first position, while each number $1 \le i < n$ has moved to position $i+1$, thus, in cycle notation $\tau = (1\ 2\ \dots\ n)$. Recall that since τ is an n-cycle, it can be written as a product of $n-1$ transpositions. Hence, $\tau \in A_n$ when n is odd, and $\tau \notin A_n$ when n is even. Notice also that with n tiles, if we continually translate in the same direction n times we should be back to our starting position. Hence $\tau^n = (1)$.

Meanwhile, the flip serves to reverse all of the numbers in the turntable, hence

$$\phi\left(\textcircled{1}\cdots\textcircled{k}\right)\textcircled{k+1}\cdots\textcircled{n} = \left(\textcircled{k}\cdots\textcircled{1}\right)\textcircled{k+1}\cdots\textcircled{n}.$$

Observe that this simultaneously swaps $\lfloor\frac{k}{2}\rfloor$ pairs of tiles, 1 and k, 2 and $k-1$, 3 and $k-2$ and so on. Thus, when k is even, we have $\phi = (1\ k)(2\ k-1)\dots(\frac{k}{2}\ \frac{k}{2}+1)$ and when k is odd, we have $\phi = (1\ k)(2\ k-1)\dots(\frac{k-1}{2}\ \frac{k+3}{2})$, as $\frac{k+1}{2}$ will be fixed in the middle of the turntable. Moreover, notice that the number of transpositions, $\lfloor\frac{k}{2}\rfloor$, is even (meaning $\phi \in A_n$) when $k \equiv 0, 1 \pmod 4$ and is odd (meaning $\phi \notin A_n$) when $k \equiv 2, 3 \pmod 4$. Certainly, if we were to immediately flip the turntable again, then we would be back to where we started. Hence $\phi^2 = (1)$.

With our two basic moves defined, we define the oval track group $OT_{n,k}$ for any $n \in \mathbb{N}$ and any integer $1 \le k \le n$ as the subgroup of S_n generated by τ and ϕ. Hence, $OT_{n,k} = \langle \tau, \phi \rangle \le S_n$ is the group, we would like to describe in general.

Degenerate cases

To start, we deal with some degenerate boundary cases. We think of these as cases where either there is something broken about the puzzle or there isn't enough room for the full structure of $OT_{n,k}$ to be realized.

Proposition 2 (Degenerate Cases)**.**

(i) $OT_{n,1} \cong \mathbb{Z}_n$,

(ii) $OT_{2,2} \cong S_2$,

(iii) For $n \geq 3$, $OT_{n,n} \cong OT_{n,n-1} \cong D_n$, the symmetries of a regular n-gon.

Proof. For (i), we have $k = 1$ and thus, our turntable does not actually accomplish anything, i.e., $\phi = (1)$. Because of this, we are left with a single generator τ of order n. For (ii), $\phi = \tau = (1\ 2)$ which generates S_2. For (iii), we will show that the oval track group has the same presentation as D_n in terms of generators and relations. Recall that the dihedral group D_n can be presented as the group generated by a reflection h and a rotation t subject to the relations $h^2 = \text{Id}$, $t^n = \text{Id}$, and $hth = t^{-1}$. We have already seen for the oval track group that $\phi^2 = (1)$ and $\tau^n = (1)$, so we need only to show that $\phi\tau\phi = \tau^{-1}$ to complete the proof. Since $\phi = \phi^{-1}$, the left side is conjugation of τ by ϕ. Thus, by Lemma 1, when $k = n$, we have

$$\phi\tau\phi^{-1} = (\phi(1)\ \phi(2)\ \ldots\ \phi(n)) = (n\ n-1\ \ldots\ 1) = \tau^{-1},$$

and similarly, when $k = n - 1$, we have

$$\phi\tau\phi^{-1} = (\phi(1)\ \phi(2)\ \ldots\ \phi(n-1)\ \phi(n)) = (n-1\ n-2\ \ldots\ 1\ n) = \tau^{-1},$$

since ϕ fixes n which is outside the turntable. ■

In terms of fixing broken puzzles, this means that if you have a puzzle with $k = 1$, it can only be fixed by returning the tiles in consecutive order clockwise around the track with any tile in the first position (or, equivalently, starting with 1 in any position). In the case when $n = k = 2$, the tiles can be replaced in either of the two possible ways. And when $n \geq 3$ and $k = n$ or $n - 1$, we can imagine using the tiles as labels for the vertices of a regular n-gon. Just as every symmetry in D_n can be completely described by giving the location of a particular label and an orientation of the labels, so too can we describe the possible ways of fixing the puzzle in this fashion. We can place any tile we choose into any position after which they must continue consecutively either clockwise or counterclockwise around the track.

Nice moves, permutations, and subgroups

Observe that our degenerate cases serve to cover all cases with $1 \leq n \leq 3$ and also any case with $k = 1$, or with $k = n$ or $n - 1$. Thus, as we move to the more general situation, we will assume for the rest of the article that $n \geq 4$ and that $1 < k < n - 1$. We would also like to note that some of the techniques that follow are due to Kaufmann and Kavountzis (see [**3**] and [**4**]), however, we found several errors in their work and thus we have chosen to reorganize and present our own arguments where our work overlaps theirs.

To give a general outline of what follows: First, we will define some helpful puzzle moves which we plan to use heavily. Second, we will use those moves to generate useful permutations, such as consecutive 3-cycles. And third, we will use those permutations together with the structure of ϕ and τ to find "subgroup bounds" on $OT_{n,k}$ inside S_n. As we saw earlier, features of our generators depend on the parity of n and on k modulo 4, thus the structures of our nice moves and the work that follows will require us to split into various subcases.

Flip-translations and shuffles. We will call $\rho = \phi\tau$ a *flip-translation*. The purpose of a flip-translation is essentially to move tiles from one side of the turntable to the other. More specifically, if we imagine putting the left-most $k-1$ tiles in the turntable into a block together, then ρ has the effect of moving the first tile to the left of the turntable to the right-side of the turntable while simultaneously reversing the order of the tiles in our block:

$$\rho\ \left(\boxed{\textcircled{1}\cdots\textcircled{k-1}}\ \textcircled{k}\right)\textcircled{k+1}\cdots\textcircled{n} = \left(\boxed{\textcircled{k-1}\cdots\textcircled{1}}\ \textcircled{n}\right)\textcircled{k}\cdots\textcircled{n-1}.$$

Similarly, ρ^{-1} allows us to move tiles in the other direction and certainly, if we apply an even number of flip-translations (or inverse flip-translations), then the block will be back into its usual (clockwise) order.

Another interesting and useful move is $\pi = \tau\phi\tau^{-1}\phi = \tau^2\rho^{-2}$ which we will refer to as a *shuffle*. Observe in general (recall $n \geq k+2$) that π acts as follows:

$$\pi\left(\textcircled{1}\cdots\textcircled{k}\right)\textcircled{k+1}\cdots\textcircled{n} = \left(\textcircled{k}\,\textcircled{k+1}\,\textcircled{1}\cdots\textcircled{k-2}\right)\textcircled{k-1}\,\textcircled{k+2}\cdots\textcircled{n}.$$

Hence, when k is even, this element is the $(k+1)$-cycle $(1\ 3\ 5\ \ldots\ k+1\ 2\ 4\ \ldots\ k)$ that cycles through the odds first and then the evens. When k is odd instead, this element is the pair of disjoint $\frac{k+1}{2}$-cycles $(1\ 3\ 5\ \ldots\ k)(2\ 4\ \ldots\ k+1)$, where one cycles through the odds and the other cycles through the evens separately.

From the definition of π, we can see that π^{-1} will take the two tiles at the left of the turntable and will insert them after the first tile to the right of the turntable, leaving the third tile in the turntable as the new first tile (see below):

$$\pi^{-1}\left(\textcircled{1}\cdots\textcircled{k}\right)\textcircled{k+1}\cdots\textcircled{n} = \left(\textcircled{3}\cdots\textcircled{k+1}\,\textcircled{1}\right)\textcircled{2}\,\textcircled{k+2}\cdots\textcircled{n}.$$

Lemma 2. (a) If $n-k$ is even, then $\tau\rho^{n-k}$ is the consecutive k-cycle $(1\ 2\ \ldots\ k)$.
(b) If k is even, $\pi^{\frac{k}{2}}$ is the consecutive $(k+1)$-cycle $(k+1\ k\ \ldots\ 1)$.

Proof. (a) As discussed above, the key observation is that each flip-translation moves one tile from the left of the turntable to the right. Hence, after performing this procedure $n-k$ times, we will have moved each of the tiles $n,\ n-1,\ \ldots,\ k+1$, one at a time, from the left side of the window to the right side. Moreover, since $n-k$ is even, our block will be in the proper order—see below:

$$\begin{aligned}\tau\rho^{n-k}\left(\boxed{\textcircled{1}\cdots\textcircled{k-1}}\ \textcircled{k}\right)\textcircled{k+1}\cdots\textcircled{n} &= \tau\left(\boxed{\textcircled{1}\cdots\textcircled{k-1}}\ \textcircled{k+1}\right)\textcircled{k+2}\cdots\textcircled{n}\,\textcircled{k}\\ &= \left(\textcircled{k}\ \boxed{\textcircled{1}\cdots\textcircled{k-1}}\right)\textcircled{k+1}\cdots\textcircled{n}.\end{aligned}$$

Hence, $\tau\rho^{n-k} = (1\ 2\ \ldots\ k)$ as claimed.
(b) Observe that when k is even, using π (which moves two tiles at a time) $\frac{k}{2}$ times serves to exactly move the k tiles $2, 3, \ldots, k+1$ in order to the left of 1, while leaving all other tiles fixed. Hence,

$$\pi^{k/2}\left(\textcircled{1}\cdots\textcircled{k}\right)\textcircled{k+1}\cdots\textcircled{n} = \left(\textcircled{2}\cdots\textcircled{k+1}\right)\textcircled{1}\,\textcircled{k+2}\cdots\textcircled{n}$$

and $\pi^{k/2} = (k+1\ k\ \ldots\ 1)$ as claimed. ∎

Creating 3-cycles. With flip-translations, shuffles, and some k and $(k+1)$-cycles at our disposal, we now create 3-cycles of various varieties. Recall that consecutive 3-cycles will allow us to generate A_n by Proposition 1.

Lemma 3. (a) If $n-k$ is even, $\pi^{-1}\left(\tau\rho^{n-k}\right)^2$ is the 3-cycle $(k-1\ k\ k+1)$.
(b) If k is even, $\pi^{k/2}\tau\pi^{k/2}\phi\rho^{-1}$ is the 3-cycle $(k\ k+1\ k+2)$.

Proof. (a) We use the inverse shuffle, π^{-1}, and the k-cycle $\tau\rho^{n-k}$ from Lemma 2 since $n-k$ is even.

$$\begin{aligned}
&\pi^{-1}\left(\tau\rho^{n-k}\right)^2\left(\textcircled{1}\dots\textcircled{k}\right)\textcircled{k+1}\dots\textcircled{n}\\
&=\pi^{-1}\left(\tau\rho^{n-k}\right)\left(\textcircled{k}\,\textcircled{1}\dots\textcircled{k-1}\right)\textcircled{k+1}\dots\textcircled{n}\\
&=\pi^{-1}\left(\textcircled{k-1}\,\textcircled{k}\,\textcircled{1}\dots\textcircled{k-2}\right)\textcircled{k+1}\dots\textcircled{n}\\
&=\left(\textcircled{1}\dots\textcircled{k-2}\,\textcircled{k+1}\,\textcircled{k-1}\right)\textcircled{k}\,\textcircled{k+2}\dots\textcircled{n}.
\end{aligned}$$

Observe that all tiles are in their original positions, except $k-1$ (now in position k), k (now in position $k+1$), and $k+1$ (now in position $k-1$).
(b) Here, we use the $(k+1)$-cycle $\pi^{k/2}$ from Lemma 2, since k is even.

$$\begin{aligned}
&\pi^{k/2}\tau\pi^{k/2}\phi\rho^{-1}\left(\textcircled{1}\dots\textcircled{k}\right)\textcircled{k+1}\dots\textcircled{n}\\
&=\pi^{k/2}\tau\pi^{k/2}\phi\left(\textcircled{k-1}\dots\textcircled{1}\,\textcircled{k+1}\right)\textcircled{k+2}\dots\textcircled{n}\,\textcircled{k}\\
&=\pi^{k/2}\tau\pi^{k/2}\left(\textcircled{k+1}\,\textcircled{1}\dots\textcircled{k-1}\right)\textcircled{k+2}\dots\textcircled{n}\,\textcircled{k}\\
&=\pi^{k/2}\tau\left(\textcircled{1}\dots\textcircled{k-1}\,\textcircled{k+2}\right)\textcircled{k+1}\,\textcircled{k+3}\dots\textcircled{n}\,\textcircled{k}\\
&=\pi^{k/2}\left(\textcircled{k}\,\textcircled{1}\dots\textcircled{k-1}\right)\textcircled{k+2}\,\textcircled{k+1}\,\textcircled{k+3}\dots\textcircled{n}\\
&=\left(\textcircled{1}\dots\textcircled{k-1}\,\textcircled{k+2}\right)\textcircled{k}\,\textcircled{k+1}\,\textcircled{k+3}\dots\textcircled{n}.
\end{aligned}$$

Observe that all tiles are in their original positions, except k (now in position $k+1$), $k+1$ (now in position $k+2$), and $k+2$ (now in position k). ∎

Lemma 4. If k is odd, the element $\pi\tau\pi^{-1}\tau^{-1}$ is the 3-cycle $(1\ 3\ k+2)$.

Proof. The key idea here is that, $\tau\pi\tau^{-1}$ is equivalent to applying π one position to the right, hence $\tau\pi\tau^{-1}$ is the pair of disjoint $\frac{k+1}{2}$-cycles $(2\ 4\ \dots\ k+1)(3\ 5\ \dots\ k\ k+2)$. The even portion of $\tau\pi\tau^{-1}$ is exactly the same as the even portion from π itself, while the odd part is different. Thus, if we multiply π together with $\left(\tau\pi\tau^{-1}\right)^{-1}$, we will obtain a permutation that only permutes odd tiles:

$$\begin{aligned}
&\pi\tau\pi^{-1}\tau^{-1}\left(\textcircled{1}\dots\textcircled{k}\right)\textcircled{k+1}\dots\textcircled{n}=\pi\tau\pi^{-1}\left(\textcircled{2}\dots\textcircled{k+1}\right)\textcircled{k+2}\dots\textcircled{n}\,\textcircled{1}\\
&=\pi\tau\left(\textcircled{4}\dots\textcircled{k+1}\,\textcircled{k+2}\,\textcircled{2}\right)\textcircled{3}\,\textcircled{k+3}\dots\textcircled{n}\,\textcircled{1}\\
&=\pi\left(\textcircled{1}\,\textcircled{4}\dots\textcircled{k+1}\,\textcircled{k+2}\right)\textcircled{2}\,\textcircled{3}\,\textcircled{k+3}\dots\textcircled{n}
\end{aligned}$$

$$= \left(\text{(k+2)}\ \text{(2)}\ \text{(1)}\ \text{(4)} \dots \text{(k)} \right) \text{(k+1)}\ \text{(3)}\ \text{(k+3)} \dots \text{(n)}.$$

Notice that 1 is in the third position, 3 is in the $(k+2)$-th position, and $k+2$ is in the first position, while all other tiles are in their original positions. ∎

Proposition 3. If n is even and k is odd, then the consecutive odd 3-cycle $(1\ 3\ 5)$ is in the oval track group $OT_{n,k}$.

Proof. Here, it makes much more sense to use symmetric group tools and notation rather than dealing with puzzles. First, we'll look at the case when $n \geq k+4$ (which really means $n \geq k+5$ since n is even). In this case, we let $\Gamma = \tau^3 \pi \tau^{-3}$ and observe that Γ is the pair of $\frac{k+1}{2}$-cycles $(4\ 6\ \dots\ k+3)(5\ 7\ \dots\ k+2\ k+4)$ by Lemma 1. Hence, conjugating the 3-cycle $\pi\tau\pi^{-1}\tau^{-1} = (1\ 3\ k+2)$ from Lemma 4 by Γ gives:

$$\Gamma^2(\pi\tau\pi^{-1}\tau^{-1})\Gamma^{-2} = (\Gamma^2(1)\ \Gamma^2(3)\ \Gamma^2(k+2)) = (1\ 3\ 5).$$

Unfortunately, this method only works when $n \geq k+4$, hence, we must deal with $n = k+3$ separately (notice $n = k+2$ makes no sense since n is even and k is odd). Here, our 3-cycle is $\pi\tau\pi^{-1}\tau^{-1} = (1\ 3\ n-1)$. Conjugating by τ^2, we obtain

$$\tau^2(\pi\tau\pi^{-1}\tau^{-1})\tau^{-2} = \tau^2(1\ 3\ n-1)\tau^{-2} = (\tau^2(1)\ \tau^2(3)\ \tau^2(n-1)) = (3\ 5\ 1).$$

Thus, in all cases, we have generated the consecutive odd 3-cycle $(1\ 3\ 5)$. ∎

Subgroup bounds. With all of the nice permutations we've constructed, we will now spend time putting some helpful bounds on $OT_{n,k}$ by both generating subgroups of $OT_{n,k}$ and also finding proper subgroups of S_n which $OT_{n,k}$ must live inside in the various cases. These bounding groups will make it easier to describe the group $OT_{n,k}$ in the general setting.

Lemma 5. If n and k are both even, or if n is odd, then $A_n \leq OT_{n,k}$.

Proof. Since we always have the n-cycle $\tau = (1\ 2\ \dots\ n)$, it suffices to show that we can construct a consecutive 3-cycle in each case by Proposition 1. By Lemma 3, if n and k have the same parity, we can generate $(k-1\ k\ k+1)$ and if n is odd and $k \equiv 0, 2 \pmod 4$, then we have $(k\ k+1\ k+2)$. ∎

Lemma 5 gives us a strong lower bound in these cases as now $OT_{n,k}$ must be either A_n or S_n by Lagrange's theorem. This still leaves us with the situation, where n is even and $k \equiv 1, 3 \pmod 4$. These cases are distinctly different from the others as τ is a Type II element when n is even, and ϕ is a Type I element when $k \equiv 1, 3 \pmod 4$. Thus, as our two generators are elements of the parity subgroup, PS_n, it follows that $OT_{n,k} \leq PS_n$ in these cases. As it turns out, the two potential cases here must also be separated, however, they do share some structure.

First, observe that if we have any permutation $\alpha \in OT_{n,k}$ that is Type II, then there exists a Type I permutation $\beta = \tau^{-1}\alpha \in OT_{n,k}$ such that $\alpha = \tau\beta$. In view of this observation, we can reduce the problem of describing the group $OT_{n,k}$ to one of describing only the Type I permutations, which are contained in $S_{\frac{n}{2}}^{\text{odd}} \times S_{\frac{n}{2}}^{\text{even}}$, instead.

Proposition 4. (a) If n is even and $k \equiv 1, 3 \pmod 4$, then $A_{\frac{n}{2}}^{\text{odd}} \times A_{\frac{n}{2}}^{\text{even}} \leq OT_{n,k}$.

(b) If n is even, and $k \equiv 3 \pmod 4$, then $S_{\frac{n}{2}}^{\text{odd}} \times S_{\frac{n}{2}}^{\text{even}} \leq OT_{n,k}$.

Proof. (a) Proposition 1 applied to the alternating group on only odd numbers (or even numbers) tells us that $A_{\frac{n}{2}}^{\text{odd}}$ is generated by the consecutive *odd* 3-cycles $(1\ 3\ 5), (3\ 5\ 7), \ldots, (n-5\ n-3\ n-1)$ and similarly, $A_{\frac{n}{2}}^{\text{even}}$ is generated by the consecutive even 3-cycles. By Proposition 3, we can generate the consecutive odd 3-cycle $(1\ 3\ 5)$. Thus, with repeated conjugation by τ, we obtain the complete set of both the consecutive odd 3-cycles and the consecutive even 3-cycles.
(b) Recall that when $k \equiv 3 \pmod 4$, then ϕ will have an odd number of transpositions. Observe that there must be an even number of transpositions involving only odd numbers or involving only even numbers. That portion of ϕ will be an element in $A_{\frac{n}{2}}^{\text{odd}}$ or in $A_{\frac{n}{2}}^{\text{even}}$ and thus its inverse (itself) is in $OT_{n,k}$ by part (a). Multiplying that inverse by ϕ will leave us with an odd element in either $S_{\frac{n}{2}}^{\text{odd}}$ or $S_{\frac{n}{2}}^{\text{even}}$. We can then conjugate by τ to obtain an odd element of the other variety. It follows that we can generate all of $S_{\frac{n}{2}}^{\text{odd}} \times S_{\frac{n}{2}}^{\text{even}}$ by Lagrange's theorem. ■

When n is even and $k \equiv 3 \pmod 4$, we have just seen that $OT_{n,k}$ contains all of the Type I permutations in S_n. Recall, however, that when $k \equiv 1 \pmod 4$ instead, then $\phi \in A_n$. Thus, if $\alpha \in OT_{n,k} = \langle \phi, \tau \rangle$ is a Type I permutation, then it follows that $\alpha \in S_{\frac{n}{2}}^{\text{odd}} \times S_{\frac{n}{2}}^{\text{even}} \cap A_n$ as any way of writing it as a product of ϕ's and τ's must have an even number of τ's. Hence, $OT_{n,k}$ definitely does not contain the full set of Type I permutations when $k \equiv 1 \pmod 4$. To describe the Type I permutations that are generated, we must split this case even further.

Observe that when $k \equiv 1 \pmod 8$, $\lfloor \frac{k}{2} \rfloor$ is a number divisible by 4, hence, ϕ will be in $A_{\frac{n}{2}}^{\text{odd}} \times A_{\frac{n}{2}}^{\text{even}}$. However, if $k \equiv 5 \pmod 8$ instead, then $\lfloor \frac{k}{2} \rfloor$ is divisible by 2, but not by 4. Hence, ϕ will be in $\left(S_{\frac{n}{2}}^{\text{odd}} \times S_{\frac{n}{2}}^{\text{even}} \cap A_n\right) \setminus A_{\frac{n}{2}}^{\text{odd}} \times A_{\frac{n}{2}}^{\text{even}}$. Moreover, since n is even, $\tau^2 = (1\ 3\ \ldots\ n-1)(2\ 4\ \ldots\ n) \in A_n$ and is Type I. Each cycle is an $\frac{n}{2}$-cycle, hence $\tau^2 \in A_{\frac{n}{2}}^{\text{odd}} \times A_{\frac{n}{2}}^{\text{even}}$ if and only if $n \equiv 2 \pmod 4$.

Proposition 5. If $n \equiv 0 \pmod 4$ or $k \equiv 5 \pmod 8$, then $S_{\frac{n}{2}}^{\text{odd}} \times S_{\frac{n}{2}}^{\text{even}} \cap A_n$ is a subgroup of $OT_{n,k}$.

Proof. Observe that $A_{\frac{n}{2}}^{\text{odd}} \times A_{\frac{n}{2}}^{\text{even}}$ makes up exactly half of $S_{\frac{n}{2}}^{\text{odd}} \times S_{\frac{n}{2}}^{\text{even}} \cap A_n$ and that we already know $A_{\frac{n}{2}}^{\text{odd}} \times A_{\frac{n}{2}}^{\text{even}} \leq OT_{n,k}$ by Proposition 4. Thus, by Lagrange's theorem, it suffices to show that we can generate one element in $S_{\frac{n}{2}}^{\text{odd}} \times S_{\frac{n}{2}}^{\text{even}} \cap A_n$ where both parts are odd permutations. Observe that when $n \equiv 0 \pmod 4$, τ^2 is a permutation in $S_{\frac{n}{2}}^{\text{odd}} \times S_{\frac{n}{2}}^{\text{even}} \cap A_n$ where both parts are odd permutations. Similarly, when $k \equiv 5 \pmod 8$, ϕ is such a permutation as well. ■

Proposition 6. If $n \equiv 2 \pmod 4$, $k \equiv 1 \pmod 8$, and $\alpha \in OT_{n,k}$ is a Type I permutation, then $\alpha \in A_{\frac{n}{2}}^{\text{odd}} \times A_{\frac{n}{2}}^{\text{even}}$.

Proof. First, observe that if α is Type I, then $\tau\alpha\tau^{-1}$ will also be Type I. Moreover, we know that α can be written as a product of ϕ's and τ's. We now prove the claim by induction on the number of terms in the product. Since τ is Type II, the base case is when $\alpha = \phi$ which is in $A_{\frac{n}{2}}^{\text{odd}} \times A_{\frac{n}{2}}^{\text{even}}$.

Now, suppose that for any $\alpha = \alpha_1\alpha_2 \cdots \alpha_t$ with $t \geq 1$ and each $\alpha_i = \phi$ or τ, we have $\alpha \in A_{\frac{n}{2}}^{\text{odd}} \times A_{\frac{n}{2}}^{\text{even}}$. Consider the case when α is Type I and has minimal presentation

$\alpha_1 \cdots \alpha_{t+1}$. If $\alpha_i = \tau$ for all i, then it follows that $t+1$ is even and that $\alpha \in A^{\text{odd}}_{\frac{n}{2}} \times A^{\text{even}}_{\frac{n}{2}}$ since $\tau^2 \in A^{\text{odd}}_{\frac{n}{2}} \times A^{\text{even}}_{\frac{n}{2}}$. If not, then let s be minimal such that $\alpha_{t+1-s} = \phi$. Hence, $\alpha = \alpha_1 \ldots \alpha_{t-s}\phi\tau^{\pm s}$. As these two cases are similar, we'll deal only with the case when the exponent on τ is positive for the sake of clarity. We may conjugate s times by τ, to get ϕ to be the rightmost element, and then multiply by ϕ, to obtain the Type I permutation

$$\gamma = \tau^s \alpha \tau^{-s} \phi = \tau^s \alpha_1 \cdots \alpha_{t-s} \phi\phi = \tau^s \alpha_1 \cdots \alpha_{t-s}.$$

This product has only t terms, thus it follows by induction that $\gamma \in A^{\text{odd}}_{\frac{n}{2}} \times A^{\text{even}}_{\frac{n}{2}}$. Now, as $\alpha = \tau^{-s}(\gamma\phi)\tau^s$ it follows that $\alpha \in A^{\text{odd}}_{\frac{n}{2}} \times A^{\text{even}}_{\frac{n}{2}}$, completing the induction. ■

Describing $OT_{n,k}$ and fixing puzzles

We now describe the oval track group $OT_{n,k}$ in each case and follow up with an interpretation of each group as a set of instructions to fix any broken puzzles. Since the oval track group elements exactly correspond to the *solvable* puzzle positions, we can fix any broken puzzle by replacing the tiles using appropriate permutations.

The Main Theorem. *If* $n \geq 4$ *and* $1 < k < n-1$, *then*

(1) $OT_{n,k} \cong S_n$ *if n is even and k is even, or if n is odd and* $k \equiv 2, 3 \pmod 4$.

(2) $OT_{n,k} \cong A_n$ *if n is odd and* $k \equiv 0, 1 \pmod 4$.

(3) $OT_{n,k} \cong PS_n$ *if n is even and* $k \equiv 3 \pmod 4$.

(4) $OT_{n,k} \cong \{\alpha, \tau\alpha \mid \alpha \in S^{odd}_{\frac{n}{2}} \times S^{even}_{\frac{n}{2}} \cap A_n\}$ *if n is even and* $k \equiv 5 \pmod 8$, *or if* $n \equiv 0 \pmod 4$ *and* $k \equiv 1 \pmod 8$.

(5) $OT_{n,k} \cong \{\alpha, \tau\alpha \mid \alpha \in A^{odd}_{\frac{n}{2}} \times A^{even}_{\frac{n}{2}}\}$ *if* $n \equiv 2 \pmod 4$ *and* $k \equiv 1 \pmod 8$.

Proof. For (1) and (2), Lemma 5 tells us that $A_n \leq OT_{n,k}$. Recall that when n is even, τ is an odd permutation and when $k \equiv 2, 3 \pmod 4$, then ϕ is an odd permutation. Hence, in each case of (1), we can generate all of S_n by Lagrange's theorem. Meanwhile, for the cases in (2), both generators are contained in A_n and thus it follows that $OT_{n,k} \cong A_n$.

For (3)–(5) recall that we have already seen that $A^{\text{odd}}_{\frac{n}{2}} \times A^{\text{even}}_{\frac{n}{2}} \leq OT_{n,k} \leq PS_n$. Moreover, we know that the Type II portion of $OT_{n,k}$ is exactly the coset of the Type I portion generated by τ. The results now follow by Propositions 4–6, respectively. ■

Recall that the set of all potential puzzle arrangements exactly corresponds to the permutations in S_n. Thus, for cases in collection (1), it does not matter how the tiles are returned when fixing the puzzle. If instead, you are in collection (2), then you must be more careful. One way to proceed here is to build cycles. Start by placing any tile anywhere in the puzzle. Then whatever location you filled, pick the corresponding tile up next (i.e. the tile that would inhabit that position in the solved puzzle) and place it anywhere. Continue this procedure until you fill the location of a tile you've already picked up (which will necessarily be the location corresponding to the first tile you picked up), this creates one cycle. Then pick up another tile you haven't placed yet and continue the process.

Since we are building a permutation out of cycles, we can determine whether the puzzle will be solvable by seeing if we can decompose those cycles into an even number of transpositions. Recall that a cycle with ℓ tiles can be decomposed in terms of $\ell - 1$ transpositions. Thus, when decomposing all of the disjoint cycles created (which together cover all n tiles), we will arrive at $n - c$ transpositions, where c is the number of cycles. Since n is odd, this number will be even (and hence the puzzle will be fixed) if and only if c is odd.

For collection (3), you can replace the tiles, however, you would like as long as tiles with the same parity never end up next to each other. Another way to think about this is to color the odd positions blue and the even positions red. Then, you can replace the tiles in any way that places the odd tiles in one color and the even tiles in the other.

For collections (4) and (5), we must try to combine the instructions for (2) and (3). We can create cycles again, but we need to be more careful in their construction. The difference is that we also need to separate tiles of different parities. One way to accomplish this is to color odd tiles blue and even tiles red and then renumber the tiles of each color from 1 to $\frac{n}{2}$ (in increasing order). In addition, we must mentally separate the locations by parity (coloring them blue and red again) and then imagine renumbering each half of the track from 1 to $\frac{n}{2}$ as well. We then assign a half of the track to each pile and create cycles as before, but from one pile at a time. This gives us an alternating blue and red track and corresponding piles of blue and red tiles numbered from 1 to $\frac{n}{2}$ to work with. If we swap colors (putting red tiles in blue locations and vice versa), then this corresponds to creating a Type II permutation. Since Type II permutations have the form $\tau\alpha$ for some Type I α, we account for this by shifting the position labels so that the first odd (blue) position is in position three.

Using that method of construction, we may satisfy the conditions of collection (4), by having an odd number of cycles in total. One possible example in this collection would be a puzzle with $n = 20$ and $k = 5$. Using blue and red tiles and positions each renumbered 1 through 10 instead of $1, 3, \ldots, 19$, we then create cycles out of each color. If we choose to place blue tiles in red positions and vice versa, then the blue positions labeled 1 through 10 correspond to the old positions $3, 5, \ldots, 19, 1$, respectively. In blue, we could make the permutation $(5\,6\,7\,10)(2\,4\,1\,3)(8)(9)$ (which is four cycles), and in red we could make $(3\,4)(2)(1\,5)(6)(7\,10\,9)(8)$ (six cycles), for a total of 10 cycles. In translating back to evens and odds, recall that we also swapped all evens with all odds, thus obtaining $(1\,6\,9\,12\,13\,20\,19\,10\,3\,8\,7\,2\,11\,14)(4\,5)(15\,16\,17\,18)$ which is an odd, Type II permutation (whose Type I part is in A_n) as desired. See Figure 2 for the construction and corresponding puzzle.

To compare, for collection (5), we actually need an odd number of cycles for each individual pile/color. The smallest such puzzle is one with $n = 14$ and $k = 9$, where we could make odds red and evens blue (thus, building a Type I permutation, since we did not swap the colors). The red permutation $(1\,4\,2\,5)(3\,7)(6)$ together with the blue

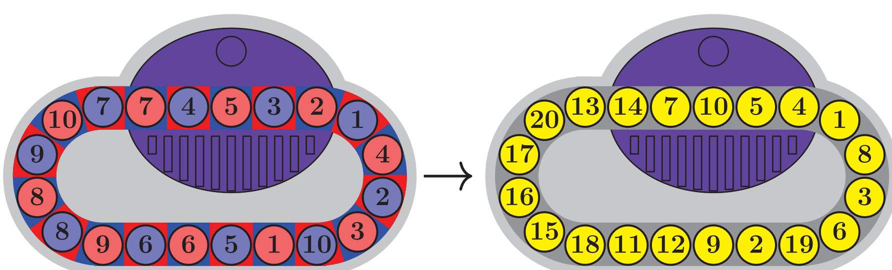
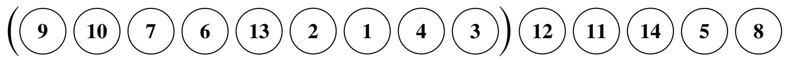


Figure 2 A possible fix for a puzzle with $n = 20$ and $k = 5$.

permutation (1 3 2 4 7 6 5) corresponds to the solvable puzzle below.



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Summary. The oval track group, $OT_{n,k}$, is the subgroup of the symmetric group, S_n generated by the basic moves in a generalized oval track puzzle with n tiles and a turntable of size k. In this article, we completely describe the oval track group for all possible n and k and use this information to answer the following question: If the tiles are removed from an oval track puzzle, how must they be returned in order to ensure that the puzzle is still solvable? As part of this discussion, we introduce the parity subgroup of S_n in the case when n is even.

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Polynomial Extensions of a Putnam Delight

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The *floor* and *ceiling* functions f and g are extremely useful in number theory, and hence in discrete mathematics. They are defined by $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$, where $\lfloor x \rfloor$ denotes the largest integer $\leq x$ and $\lceil x \rceil$ the smallest integer $\geq x$, and x is an arbitrary real number. Both notations were introduced by K.E. Iverson in the early 1960s, and are variations of the original greatest integer notation $[x]$ [2]. Both functions satisfy several interesting properties. For example, $\lfloor n \rfloor = n = \lceil n \rceil$; $\lfloor x + n \rfloor = \lfloor x \rfloor + n$; and $\lceil x + n \rceil = \lceil x \rceil + n$, where x is a real number and n is an integer.

The following charming problem appeared in the 68th Annual William Putnam Mathematical Competition in 2007:

> Find an explicit formula for x_n, where $x_{n+1} = 3x_n + \lfloor \sqrt{5} x_n \rfloor$, $x_0 = 1$, $n \geq 0$.

Surprisingly, its solution leads to a class of the well-known Fibonacci numbers F_n. The given recurrence, together with induction, can be employed to establish that $x_n = 2^{n-1} F_{2n+3}$, where $n \geq 0$ [1,4]. We encourage readers to construct a proof of their own for this problem. Lucas, Pell, and Pell-Lucas counterparts appeared in [5].

Fibonacci and Lucas numbers, and their polynomial extensions continue to be a fertile ground for creativity and exploration, and a source of fun and excitement. So are their close relatives, Pell and Pell-Lucas numbers and polynomials. In this article, we investigate polynomial extensions of the Putnam delight. To this end, we first introduce the family of Gibonacci polynomials; it includes Fibonacci, Lucas, Pell, and Pell-Lucas polynomials as subfamilies.

Gibonacci polynomials

Gibonacci polynomials $g_n(x)$ satisfy the recurrence $g_n(x) = x g_{n-1}(x) + g_{n-2}(x)$, where $g_1(x) = a = a(x)$ and $g_2(x) = b = b(x)$ are arbitrary polynomials, and $n \geq 3$. We can extend this recurrence to include the case $n = 0$ by letting $g_0(x) = b - ax$. When $a = 1$ and $b = x$, $g_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $a = x$ and $b = x^2 + 2$, $g_n(x) = l_n(x)$, the nth *Lucas polynomial*. In particular, $g_n(1) = G_n$, the nth *Gibonacci number*; $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [3,4]. Table 1 shows the first six Fibonacci and Lucas polynomials.

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1)$ and $2Q_n = q_n(1)$, respectively [3,4].

Math. Mag. **91** (2018) 359–362. doi:10.1080/0025570X.2018.1507400
MSC: Primary 11AXX, Secondary 11B37; 11B39

n	$f_n(x)$	$l_n(x)$
1	1	x
2	x	x^2+2
3	x^2+1	x^3+3x
4	x^3+2x	x^4+4x^2+2
5	x^4+3x^2+1	x^5+5x^3+5x
6	x^5+4x^3+3x	$x^6+6x^4+9x^2+2$

TABLE 1: First six Fibonacci and Lucas polynomials.

In the interest of brevity and convenience, we will omit the argument in the functional notation; so g_n will mean $g_n(x)$, etc.

Binet-like formulas Fibonacci and Lucas polynomials can also be defined explicitly by the *Binet-like formulas*

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad l_n = \alpha^n + \beta^n,$$

where $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are solutions of the characteristic equation $t^2 - xt - 1 = 0$, and $n \geq 0$ [3, 4]. These formulas can be confirmed by solving the recurrences or using induction.

Notice that $2\alpha = x + \Delta, 2\beta = x - \Delta$, and $2\alpha^2 = x^2 + x\Delta + 2$, where $\Delta = \Delta(x) = \sqrt{x^2+4}$. We will employ these properties later.

Pell and Pell-Lucas polynomials can be defined by Binet-like formulas

$$p_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad q_n = \gamma^n + \delta^n,$$

where $\gamma = \gamma(x)$ and $\delta = \delta(x)$ are solutions of the characteristic equation $t^2 - 2xt - 1 = 0$, and $n \geq 0$ [3, 4]. Again, both formulas can be confirmed by solving the recurrences or using induction.

With these tools at hand, we are now ready for the Fibonacci, Lucas, Pell, and Pell-Lucas polynomial extensions of the Putnam problem. To begin with, we focus on the Fibonacci polynomial extension.

Fibonacci polynomial extension

Find an explicit formula for x_n, where $x_{n+1} = (x^2+2)x_n + \lfloor x\Delta x_n \rfloor$, $x_0 = \dfrac{x^2+1}{2}$, $n \geq 0$, and x is a positive integer.

We will establish that $x_n = 2^{n-1} f_{2n+3}$.

Proof. We will confirm this statement using induction. Clearly, it is true when $n = 0$. Now assume it is true for an arbitrary nonnegative integer n. Then

$$\begin{aligned} x_{n+1} &= (x^2+2)x_n + \lfloor x\Delta x_n \rfloor = \lfloor (x^2 + x\Delta + 2)x_n \rfloor = \lfloor 2\alpha^2 x_n \rfloor \\ &= \lfloor 2\alpha^2 \cdot 2^{n-1} f_{2n+3} \rfloor = \left\lfloor 2^n \alpha^2 \cdot \frac{\alpha^{2n+3} - \beta^{2n+3}}{\alpha - \beta} \right\rfloor \end{aligned}$$

$$= \left\lfloor \frac{2^n}{\alpha - \beta}\left[(\alpha^{2n+5} - \beta^{2n+5}) + (\beta^{2n+5} - \beta^{2n+1})\right]\right\rfloor$$

$$= 2^n f_{2n+5} + \left\lfloor \frac{2^n \beta^{2n+3}}{\alpha - \beta}(\beta^2 - \alpha^2)\right\rfloor = 2^n f_{2n+5} + \left\lfloor -2^n x\beta^{2n+3}\right\rfloor.$$

The desired result will now follow if $\lfloor -2^n x\beta^{2n+3}\rfloor = 0$. We will show that it is so.

Since $2\alpha^2 = x^2 + x\Delta + 2 > 1 + 1 + 2 = 4, \alpha^2 > 2$. Consequently, $0 < 2\beta^2 = 2/\alpha^2 < 1$, and hence $0 < \beta^2 < 1$. We also have

$$-2\beta = \Delta - x = \frac{\Delta^2 - x^2}{\Delta + x} = \frac{4}{\Delta + x} < \frac{4}{x + x} = \frac{2}{x};$$

hence, $0 < -\beta x < 1$. Then, $0 < (2\beta^2)^n \beta^2(-\beta x) < 1$ and $\lfloor -2^n x\beta^{2n+3}\rfloor = 0$.

Thus, $x_{n+1} = 2^n f_{2n+5}$, so the formula works for $n + 1$ also. Consequently, the formula works for all $n \geq 0$, as desired. ■

For example, $x_5 = (x^2 + 2)x_4 + \lfloor x\Delta x_4\rfloor = 2^4 f_{13} = 16(x^{12} + 11x^{10} + 45x^8 + 84x^6 + 70x^4 + 21x^2 + 1)$.

Clearly, when $x = 1$, the formula $x_n = 2^{n-1} f_{2n+3}$ yields the solution to the original Putnam problem [1,4,5].

Next we study the Lucas extension of the problem.

Lucas polynomial extension

> Find an explicit formula for x_n, where $x_{n+1} = (x^2 + 2)x_n + \lceil x\Delta x_n\rceil$, $x_0 = \dfrac{x^3 + 3x}{2}$, $n \geq 0$, and x is a positive integer.

We will prove that $x_n = 2^{n-1} l_{2n+3}$.

Proof. Again, we invoke induction to confirm the formula. Since $x_0 = 2^{-1} l_3$, the formula works when $n = 0$.

Assume it is true for an arbitrary nonnegative integer n. Then

$$x_{n+1} = (x^2 + 2)x_n + \lceil x\Delta x_n\rceil = \lceil (x^2 + x\Delta + 2)x_n\rceil = \left\lceil 2\alpha^2 x_n\right\rceil$$

$$= \left\lceil 2\alpha^2 \cdot 2^{n-1} l_{2n+3}\right\rceil = \left\lceil 2^n \alpha^2\left(\alpha^{2n+3} + \beta^{2n+3}\right)\right\rceil$$

$$= \left\lceil 2^n\left[(\alpha^{2n+5} + \beta^{2n+5}) - (\beta^{2n+5} - \beta^{2n+1})\right]\right\rceil$$

$$= 2^n l_{2n+5} + \left\lceil 2^n \beta^{2n+3}(\alpha^2 - \beta^2)\right\rceil = 2^n l_{2n+5} + \left\lceil 2^n x\Delta\beta^{2n+3}\right\rceil.$$

We will now show that $\lceil 2^n x\Delta\beta^{2n+3}\rceil = 0$. To this end, first notice that

$$x\Delta\beta^2 = x\Delta \cdot \frac{(x-\Delta)^2}{4} = \frac{x\Delta}{4} \cdot \frac{(x^2 - \Delta^2)^2}{(x + \Delta)^2} = \frac{x\Delta}{4} \cdot \frac{(-4)^2}{(x + \Delta)^2} = \frac{4x\Delta}{(x + \Delta)^2} < 1;$$

and

$$\beta^2 = \left(\frac{x - \Delta}{2}\right)^2 = \left[\frac{\Delta^2 - x^2}{2(\Delta + x)}\right]^2 = \frac{4}{(\Delta + x)^2} \leq \frac{4}{(1 + \sqrt{5})^2} < \frac{1}{2}.$$

It now follows by the above inequalities that $x\Delta\beta^2 \cdot (\beta^2)^n < (1/2)^n$; that is, $2^n x\Delta\beta^{2n+2} < 1$. Since $-1 < \beta < 0$, this implies $-1 < 2^n x\Delta\beta^{2n+3} < 0$; so $\lceil 2^n x\Delta\beta^{2n+3}\rceil = 0$.

Thus ,$x_{n+1} = 2^n l_{2n+5}$. Hence, the formula works for $n+1$ as well, and by induction, it works for all integers $n \geq 0$, as desired. ■

For example, $x_4 = (x^2+2)x_3 + \lceil x\Delta x_3 \rceil = 2^3 l_{11} = 8(x^{11} + 11x^9 + 44x^7 + 77x^5 + 55x^3 + 11x)$.

In particular, consider the recurrence $x_{n+1} = 3x_n + \lceil \sqrt{5} x_n \rceil$, where $x_0 = 2$ and $n \geq 0$. Then, $x_n = 2^{n-1} L_{2n+3}$ [4,5]. As an example, $x_5 = 3x_4 + \lceil \sqrt{5} x_4 \rceil = 3(2^3 L_9) + \lceil \sqrt{5}(2^3 L_9) \rceil = 24 \cdot 76 + \lceil \sqrt{5}(8 \cdot 76) \rceil = 3184 = 2^4 L_{11}$, as expected.

Interestingly, the Fibonacci and Lucas versions have their own Pell and Pell-Lucas counterparts.

Pell and Pell-Lucas counterparts

> Consider the sequence $\{x_n\}$, where $x_{n+1} = (2x^2+1)x_n + \lceil 2x\sqrt{x^2+1}x_n \rceil$, x is a positive integer, $n \geq 0$, and $x_0 = 2x$. Then, $x_n = p_{2n+2}$. On the other hand, if $x_{n+1} = (2x^2+1)x_n + \lfloor 2x\sqrt{x^2+1}x_n \rfloor$ and $x_0 = 4x^2+2$, then $x_n = q_{2n+2}$.

The proofs follow using similar steps; so we omit them in the interest of brevity. But we encourage Fibonacci and Lucas enthusiasts to confirm both formulas.

In particular, let $x = 1$. Then, the recurrence $x_{n+1} = 3x_n + \lceil 2\sqrt{2} x_n \rceil$ with the initial condition $x_0 = 2$, implies $x_n = P_{2n+2}$; likewise, the recurrence $x_{n+1} = 3x_n + \lfloor 2\sqrt{2} x_n \rfloor$, coupled with $x_0 = 6$, yields $x_n = 2Q_{2n+2}$ [5].

For example, consider the latter recurrence with $x_0 = 6$. Then, $x_{10} = 3x_9 + \lceil 2\sqrt{2} x_9 \rceil = 3(2Q_{20}) + \lceil 2\sqrt{2}(2Q_{20}) \rceil = 24 \cdot 22{,}619{,}537 + \lceil 4\sqrt{2} \cdot 22{,}619{,}537 \rceil = 2 \cdot 131{,}836{,}323 = 2Q_{22}$, as desired.

Acknowledgment The authors thank the Editor and reviewers for their constructive suggestions and helpful comments for the improvement of the exposition of the original version.

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Summary. Fibonacci, Lucas, Pell, and Pell-Lucas polynomials are a fertile ground for imagination and creativity. They offer boundless exploratory opportunities for Fibonacci and Lucas enthusiasts. This article features one such activity; it extends a Fibonacci problem that appeared in the 68th William Putnam Mathematical Competition to Fibonacci, Lucas, Pell, and Pell-Lucas polynomials.

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Proof Without Words: Tangent Plus Cotangent is Greater or Equal Than 2

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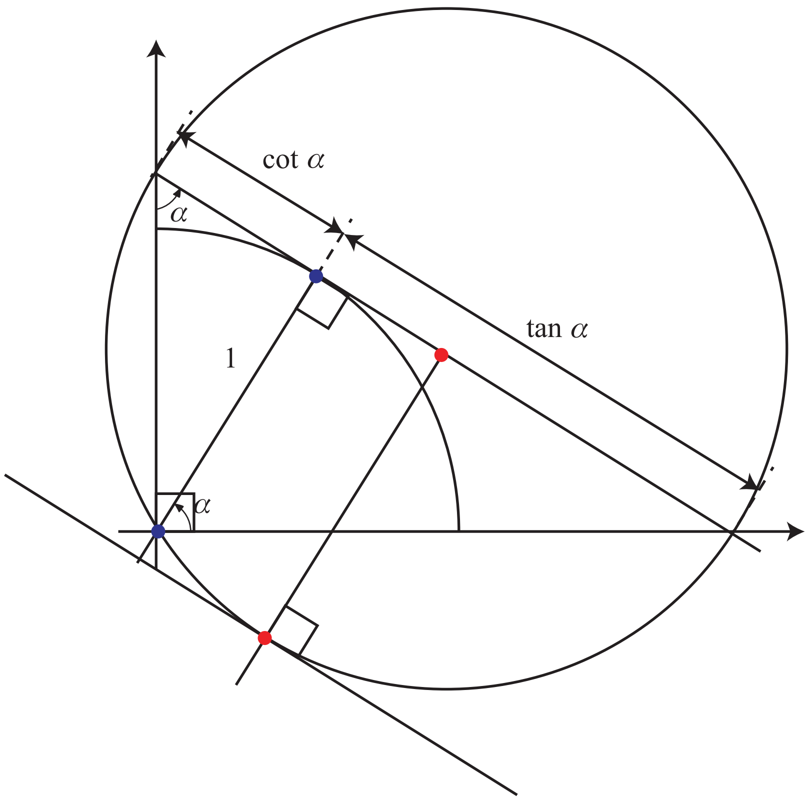
Theorem. *For* $\alpha \in (0, \pi/2)$, $\tan\alpha + \cot\alpha \geq 2$.

Proof.



■

Corollary. *For* $x > 0$, $x + \dfrac{1}{x} \geq 2$.

Summary. Using the representation of $\tan\alpha$ and $\cot\alpha$ in the unit circle, it is shown without words that $\tan\alpha + \cot\alpha \geq 2$.

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Math. Mag. **91** (2018) 363. doi:10.1080/0025570X.2018.1465772
MSC: Primary 97G40
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/umma.

Periodic Continued Fractions Via a Proof Without Words

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An *infinite simple continued fraction* is an expression of the form

$$[a_0, a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

where a_0 is an integer and $a_1, a_2, a_3, \ldots$ are positive integers. When a block of a_k's repeats over and over, the continued fraction is *periodic*. For example,

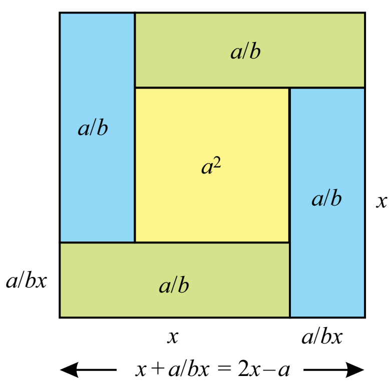
$$[\overline{a}] = [a, a, a, \ldots] = a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cdots}}}$$

$$\text{and } [\overline{a, b}] = [a, b, a, b, \ldots] = a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \cdots}}}$$

are periodic (just like in repeating decimals, the vinculum indicates the repeating block). We find a closed form expression for $[\overline{a, b}]$ (note that $[\overline{a}]$ is a special case of $[\overline{a, b}]$) by proving Lemma 1 wordlessly with a frequently used diagram [1]-[4], proving Lemma 2 by simple algebra, from which the theorem follows immediately.

Lemma 1. *If* $x > 0$ *and* $x = a + \frac{a}{bx}$, *then* $x = \frac{1}{2}[a + \sqrt{a^2 + 4(a/b)}]$.

Proof.



$$(2x - a)^2 = a^2 + 4(a/b)$$
$$x = \frac{1}{2}\left[a + \sqrt{a^2 + 4(a/b)}\right]$$

■

Lemma 2. *If* $x = [\overline{a, b}] = a + \frac{1}{b + \frac{1}{x}}$ *then* $x = a + \frac{a}{bx}$.

Proof. Because $x = a + \frac{1}{b+\frac{1}{x}}$, then $x\left(b + \frac{1}{x}\right) = a\left(b + \frac{1}{x}\right) + 1$. It follows that $bx = ab + \frac{a}{x}$ so that $x = a + \frac{a}{bx}$. ■

Math. Mag. **91** (2018) 364–365. doi:10.1080/0025570X.2018.1456151 © Mathematical Association of America
MSC: Primary 11A55
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Theorem. *The periodic continued fraction* $[\overline{a, b}]$ *equals* $\frac{1}{2}[a + \sqrt{a^2 + 4(a/b)}]$.

As examples, notice that $[\overline{1}] = \frac{1}{2}(1 + \sqrt{5})$, $[\overline{2}] = 1 + \sqrt{2}$, $[\overline{2, 1}] = 1 + \sqrt{3}$, and $[\overline{3, 2}] = \frac{1}{2}(3 + \sqrt{15})$.

Exercise. Show that for n a positive integer,

(a) $[n, \overline{2n}] = \sqrt{n^2 + 1}$

(b) $[n, \overline{n, 2n}] = \sqrt{n^2 + 2}$,

(c) $[n, \overline{2, 2n}] = \sqrt{n^2 + n}$, and

(d) $[n, \overline{1, 2n}] = \sqrt{n^2 + 2n}$.

As a hint, consider $[\overline{2n}]$, $[\overline{2n, n}]$, $[\overline{2n, 2}]$, and $[\overline{2n, 1}]$.

Acknowledgment The author wishes to thank two referees and the Editor for helpful suggestions on an earlier draft of this note.

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Summary. We evaluate some periodic continued fractions by computing the area of a square in two different ways.

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Picard's Weighty Proof of Chebyshev's Sum Inequality

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"Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap."—declared the eminent Russian mathematician, Vladimir Igorevich Arnold (1937–2010) in an address on teaching mathematics [3]. Although Arnold's words might sound a bit presumptuous, it is common that behind a seemingly pure mathematical concept, quite natural physical principles lie. For example, George Pólya (1887–1985) devoted a whole chapter to physical mathematics in his book on plausible reasoning [9]. Further, some years ago Mark Levi in [7] revealed dozens of surprising links between mathematics and physics, including an electrical proof of the inequality of arithmetic and harmonic means, which also appeared recently in *This Magazine* [11].

In this note, we continue along the above philosophy and focus attention on an historical and interesting physical demonstration—which dates back to the French mathematician, Émile Picard (1856–1941)—of the following familiar algebraic inequality for real numbers often referred to as Chebyshev's sum inequality (or Chebyshev's order inequality in [10]).

Theorem 1 (Chebyshev's sum inequality)**.** *If* $u_1 \le u_2 \le \cdots \le u_n$ *and* $v_1 \le v_2 \le \cdots \le v_n$ *(or both sequences are weakly decreasing), then*

$$(u_1 + u_2 + \cdots + u_n)(v_1 + v_2 + \cdots + v_n) \le n(u_1v_1 + u_2v_2 + \cdots + u_nv_n). \qquad (1)$$

As a special case when the two sequences are equal, Chebyshev's inequality becomes essentially the square of the inequality between the arithmetic mean and the root mean square.

Historical background

Chebyshev's sum inequality is named after Pafnuty Lvovich Chebyshev (1821–1894), one of the founding fathers of Russian mathematics. In a brief note [4] of 1882, he formulated the integral version of the above inequality in a rather general form and published its proof in a subsequent paper [5]. Chebyshev's general inequality implies, as a special case, that if the functions $u, v\colon [0, 1] \to \mathbb{R}$ are increasing (or simultaneously decreasing), then

$$\left(\int_0^1 u\,dx\right) \cdot \left(\int_0^1 v\,dx\right) \le \int_0^1 uv\,dx. \qquad (2)$$

Math. Mag. **91** (2018) 366–371. doi:10.1080/0025570X.2018.1512814
MSC: Primary 26D15, Secondary 05A20; 70C20

This integral inequality was communicated by Chebyshev to the French mathematician Charles Hermite (1822–1901) who then included it, with the extra assumption that u, v are nonnegative and strictly monotone, in the lecture notes [1, pp. 48–49] for his analysis course taught at the Sorbonne in the second semester of 1881–1882. Hermite gave acknowledgment to Chebyshev, and then presented a proof due to Picard. Picard's reasoning started with the reduction of the integral inequality to the discrete one, which then was placed into a physical setting based on the notion of the center of gravity. Although this proof seems lesser-known, it is one of the many spectacular encounters between mathematics and physics.

We first briefly recall the very intuitive concepts of torque (or moment), center of gravity (or center of mass in other terminology), and formulate the physical principles on which Picard relied. After reproducing Picard's arguments, we shall also present a mechanical interpretation of the classical proof of Chebyshev's inequality based on the rearrangement inequality.

The center of gravity in the center of attention

Suppose there are point particles with positive masses $m_1, \ldots, m_n$ located at coordinates $x_1, \ldots, x_n$, respectively, on the real axis, which we now consider as a weightless horizontal rod (see Figure 1).

If the system is supported from below (or suspended) at a pivot point, then the downward gravitational pull on each mass results in a clockwise or counterclockwise rotation around the point. This turning effect is the so-called torque or moment, the concept of which was already used in mathematics by Archimedes of Syracuse, when he calculated areas and volumes of various shapes with his ingenious method (see [2]). The magnitude of the torque equals the product of the weight of the particle (*i.e.,* its mass times the gravitational acceleration g) and the lever arm (*i.e.,* the distance between the particle and the pivot). Let us now work with signed torques: we assign a plus or minus sign to each torque depending on whether its rotational effect is counterclockwise or clockwise, respectively. Equivalently, the lever arm is considered as the signed distance between the particle and the pivot: it is negative when the particle is on the right side of the pivot. With this in mind, the torque about some pivot point x due to the particle with mass m_i is $m_i g(x - x_i)$. For simplicity, we can neglect the constant factor g, thus the total torque of the particles with respect to the point x becomes the sum $m_1(x - x_1) + m_2(x - x_2) + \cdots + m_n(x - x_n)$.

The center of gravity of a system of particles is the pivot point where the system is completely balanced when supported from below: the individual turning effects cancel each other and the total torque becomes zero. Thus, if the particle with mass m_i is located at x_i for all i and x_{cg} denotes the center of gravity, the condition of equilibrium takes the form

$$m_1(x_{\mathrm{cg}} - x_1) + m_2(x_{\mathrm{cg}} - x_2) + \cdots + m_n(x_{\mathrm{cg}} - x_n) = 0,$$

or, equivalently,

$$(m_1 + m_2 + \cdots + m_n)x_{\mathrm{cg}} = m_1x_1 + m_2x_2 + \cdots + m_nx_n. \tag{3}$$

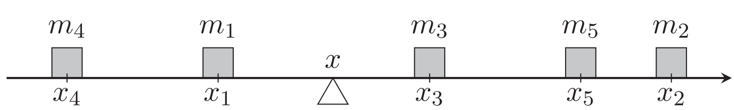

Figure 1 System of masses support from below.

Therefore the center of gravity of the system becomes

$$x_{\mathrm{cg}} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}. \tag{4}$$

In other words, the coordinate of the center of gravity is the weighted average of the coordinates of the masses. Consequently, if all the masses are equal, then the center of gravity is located at the arithmetic mean of the coordinates regardless of the particular value of the common mass.

Clearly, if a pivot point is located on the same side of each mass, then all the torques about this point act in the same direction and thus equilibrium cannot be achieved. This yields an important principle.

Principle 1. The center of gravity lies between the leftmost and rightmost of the masses.

The relation (3) expresses the fact that total torque of the system about the origin is the same as if the total mass of the system were concentrated at the center of gravity. Since the origin can be chosen freely, we obtain that the total torque of a system of masses about any point is identical with the turning effect of the total mass located at the center of gravity of the system. Practically, this implies another fundamental principle.

Principle 2. The center of gravity of a system does not change when a subsystem of its masses is replaced by the total mass of the subsystem concentrated at the center of gravity of the subsystem.

We still need a third intuitive observation. By removing the leftmost mass from the system, that is the one with the smallest coordinate, the clockwise torques overcome the counterclockwise torques about the original center of gravity, therefore the new center of gravity should be located on the right side of the original one.

Principle 3. By removing the leftmost/rightmost mass from a system, the center of gravity shifts to the right/left.

It is an easy exercise to derive all the above principles in a rigorous mathematical way by using the particular formula (4) for the center of gravity. Now we are ready to present Picard's arguments.

Picard's proof rephrased

By following Hermite and Picard, we first add the extra assumption to Chebyshev's inequality that the sequences are positive and strictly monotone. For aesthetic reasons, we rephrase Picard's proof in the case when the two sequences are simultaneously increasing. Let us start by rewriting inequality (1) in the more suggestive form

$$\frac{u_1 + u_2 + \cdots + u_n}{n} \le \frac{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}{v_1 + v_2 + \cdots + v_n}. \tag{5}$$

We now consider the points $A_1, A_2, \ldots, A_{n-1}, A_n$ on the real axis with coordinates $u_1 \le u_2 \le \cdots \le u_{n-1} \le u_n$ and concentrate mass v_i at the point A_i. Then the system's center of gravity x_{cg} becomes the right-hand side of (5). In view of the telescoping sum

$$v_k = v_1 + (v_2 - v_1) + (v_3 - v_2) + \cdots + (v_{k-1} - v_{k-2}) + (v_k - v_{k-1}),$$

our system can be decomposed into some subsystems as follows. In the first subsystem we concentrate mass v_1 at each point $A_1, A_2, \ldots, A_n$; in the second subsystem mass $v_2 - v_1$ at each point $A_2, A_3, \ldots, A_n$ and so forth; in the nth system mass $v_n - v_{n-1}$ is concentrated at point A_n. For simplicity, let us denote the center of gravity of the ith subsystem by $x_{\mathrm{cg}}^{(i)}$. Observe that $x_{\mathrm{cg}}^{(1)}$ is exactly the left-hand side of (5). Now, by Principle 2, the center of gravity x_{cg} of the whole system does not change if we replace each subsystem by their total mass concentrated at $x_{\mathrm{cg}}^{(i)}$. Therefore, Principle 1 implies that x_{cg} is located between the two extremes of the coordinates $x_{\mathrm{cg}}^{(i)}$ $(i = 1, \ldots, n)$. We claim that $x_{\mathrm{cg}}^{(1)} \leq x_{\mathrm{cg}}^{(2)} \leq \cdots \leq x_{\mathrm{cg}}^{(n)}$, thus $x_{\mathrm{cg}}^{(1)} \leq x_{\mathrm{cg}}$, which is exactly inequality (5). The relation $x_{\mathrm{cg}}^{(k)} \leq x_{\mathrm{cg}}^{(\ell)}$ for $k < \ell$ follows by Principle 3, since a removal of the leftmost points $A_k, \ldots, A_{\ell-1}$ from the kth subsystem yields a system of equal masses concentrated at $A_\ell, \ldots, A_n$, which has the same center of gravity as the ℓth subsystem.

Looking back and ahead

It is readily seen from the proof that for strict monotone sequences the inequality is strict as well (in fact, this was proved by Picard). However, the proof applies also if we forget about the strict monotonicity condition. In case $v_k - v_{k-1} = 0$ for some (but not all) k, the omission of these subsystems will not affect the rest of the argument (and for constant (v_i) equality is evident). It is likewise admissible that some of the points A_i coincide.

Furthermore, the positivity assumption added by Hermite and Picard is also not a proper restriction. If the sequence (u_i) contains some nonnegative terms but (v_i) is positive, then we can choose some real number d such that $u_i + d > 0$ for all i (which is physically a translation of the origin). Hence

$$\big((u_1 + d) + \cdots + (u_n + d)\big)(v_1 + \cdots + v_n) \leq n\big((u_1 + d)v_1 + \cdots + (u_n + d)v_n\big)$$

from where d vanishes and inequality (1) follows. When both sequences contain non-negative terms, the same substitution can be applied successively.

If $u, v\colon [0, 1] \to \mathbb{R}$ are simultaneously increasing (or decreasing) functions, then by taking the partition $x_i = i/n$ $(i = 0, \ldots, n)$ of the interval $[0, 1]$ and letting $u_i = u(x_i)$, $v_i = v(x_i)$, Chebyshev's sum inequality implies

$$\frac{1}{n}\sum_{i=1}^{n} u\left(\frac{i}{n}\right) \cdot \frac{1}{n}\sum_{i=1}^{n} v\left(\frac{i}{n}\right) \leq \frac{1}{n}\sum_{i=1}^{n} (uv)\left(\frac{i}{n}\right).$$

From here, we obtain inequality (2) as $n \to \infty$.

A moment on rearrangement

Motivated by Picard's arguments, we now present a simple mechanical interpretation of the usual proof of Chebyshev's inequality carried out via the rearrangement inequality. To this end, assume that $x_1 \leq x_2 \leq \cdots \leq x_n$, $m_1 \leq m_2 \leq \cdots \leq m_n$ and let us allocate to each coordinate x_i exactly one particle so as to maximize the total torque of the system about the origin. Intuitively, the particle with the largest mass m_n should be associated with the largest lever arm x_n; the particle with the second largest mass to the second largest lever arm and so forth. Indeed, suppose we have a different arrangement in which j is the maximal index for which particle with mass m_j is not located at coordinate x_j but at some x_k, where $k < j$. Then some particle with mass m_ℓ is located

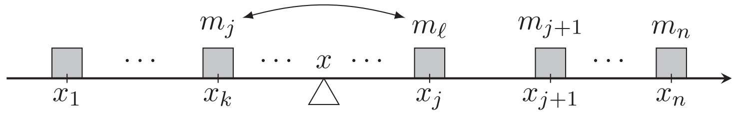

Figure 2 Swapping two masses.

at x_j where $\ell < j$ (see Figure 2). If one swaps only the particles with masses m_j and m_ℓ, then the total torque about the origin does not decrease since it changes by

$$(m_\ell x_k + m_j x_j) - (m_j x_k + m_\ell x_j) = (m_j - m_\ell)(x_j - x_k) \geq 0.$$

In the new arrangement mass m_j is placed at x_j; by repeating this procedure we obtain that the masses are in increasing order and the torque has not been decreased.

Note that the positivity of the masses is not essential in the previous argument, because we can add some mass d to each m_i to guarantee $m_i + d > 0$. Then in any arrangement the total torque increases by the amount $d(x_1 + x_2 + \cdots + x_n)$, which does not affect the order between the total torques. We have obtained the well-known rearrangement inequality [6].

Theorem 2 (Rearrangement inequality)**.** *Assume that $u_1 \leq u_2 \leq \cdots \leq u_n$ and $v_1 \leq v_2 \leq \cdots \leq v_n$. If $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n$ is any permutation of the numbers $v_1, v_2, \ldots, v_n$, then*

$$u_1\tilde{v}_1 + u_2\tilde{v}_2 + \cdots + u_n\tilde{v}_n \leq u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Chebyshev's inequality might be interpreted similarly. We consider n^2 particles: n particles with mass v_i for all i where $v_1 \leq v_2 \leq \cdots \leq v_n$. Let us place at each coordinate $u_1 \leq u_2 \leq \cdots \leq u_n$ exactly n particles in such a way that the total torque is maximal. The same argument applies as before: the maximum is attained when all the n particles with mass v_i are assigned to u_i for all i. Then the total torque becomes the right-hand side of (1) while the left-hand side of (1) is the total torque of the system when the total mass located at u_i is $v_1 + v_2 + \cdots + v_n$ for all i. We have now obtained a physical explanation of both the rearrangement inequality and Chebyshev's inequality in terms of torque.

History goes on

The third edition of Hermite's lecture notes appeared in 1887 contained neither Picard's proof nor the discrete inequality (1). Only the integral version was demonstrated based on the nice and easy to check integral identity

$$\frac{1}{2}\int_a^b \int_a^b \left[\big(f(x) - f(y)\big)\big(g(x) - g(y)\big)\right] dxdy$$
$$= (b-a)\int_a^b f(x)g(x)\,dx - \left(\int_a^b f(x)\,dx\right) \cdot \left(\int_a^b g(x)\,dx\right).$$

Although, Hermite ascribed this identity to the Hungarian born American mathematician Fabian Franklin (1853–1939), who used it in a paper of 1885, it was already established in 1883 by the Russian mathematician Konstantin Alekseevich Andreev (1848–1921), who at that time was a professor at the University of Kharkov. The discrete analogue of the identity has the form

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) + \frac{1}{n^2}\sum_{i<j}(x_i - x_j)(y_i - y_j),$$

which is due to the Russian mathematician Aleksandr Nikolayevich Korkin (1837–1908), a former student of Chebyshev, who communicated it to Hermite in a letter of 1883. Nowadays, the usual proof of Chebyshev's inequality is based on the above algebraic identity or the rearrangement inequality.

We stop here in the journey, for more details we refer to the monograph [8, Chap. IX], where the complete history and evolution of Chebyshev's inequality and its generalizations is nicely collected together with accurate bibliographic details of the above-cited papers as well. Finally, regarding many other applications of the center of gravity in mathematical demonstrations, we highly recommend [7] for the interested reader.

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Summary. In his analysis course notes of 1882, Hermite included an algebraic inequality known today as Chebyshev's sum inequality. He presented a physical demonstration due to Picard which was based on the intuitive concept of the center of gravity. We first recall Picard's reasoning with some historical background and then, motivated by his idea, we provide a mechanical interpretation of the usual proof of Chebyshev's inequality carried out via the rearrangement inequality.

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Partiti by ILP

Integer-Linear Programming (ILP) is extremely powerful and is used to solve a multitude of real-world scheduling and assignment problems (e.g., making schedules for major league sports). The Partiti puzzles, present in each 2018 issue of THIS MAGAZINE, are easily solved by ILP. If you are unfamiliar with the puzzle, the current issue's puzzle is below, including the rules.

The Partiti puzzle consists of 36 squares (say, numbered left-to-right, top-to-bottom). An ILP solution of a Partiti puzzle uses variables that list all possible partitions for the entry in square i. Suppose that the ith square has an entry of 6. Then the four variables for the ith square are $x_i[\{6\}]$, $x_i[\{1, 5\}]$, $x_i[\{2, 4\}]$ and $x_i[\{1, 2, 3\}]$. There are four variables because there are four partitions of 6 using distinct integers from $\{1, 2, \ldots, 9\}$: $6 = 1 + 5 = 2 + 4 = 1 + 2 + 3$. The idea is that these variables will lie in the interval [0, 1], but will be constrained to be integers. Two additional constraints are that, for each square, exactly one of the variables is 1 and, for any pair of neighboring squares, the partitions that yield the 1s are disjoint.

For the April 2018, Partiti puzzle [*Math. Mag.* 91(2), p. 91], the complete system can be set up quickly in *Mathematica* and has 349 variables and 10978 constraints; this approach yields the solution in about a second. For more on the code used, see the supplemental data for this note on the publisher's website.

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Partiti Puzzle

15	11	16	13	15	11
9	10	8	8	9	10
18	8	13	8	13	13
6	11	8	16	8	11
6	13	1	13	8	13
18	6	23	8	11	13

How to play. In each cell, place one or more distinct integers from 1 to 9 so that they sum to the value in the top left corner. No integer can be used more than once in horizontally, vertically, or diagonally adjacent cells. For an introduction to the Partiti Puzzle, see [Caicedo, A. E., Shelton, B. (2018). Of puzzles and partitions: Introducing Partiti. *Math. Mag.* 91(1): 20–23]. The solution is on page 385.

—contributed by Lai Van Duc Thinh,
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Euclid's Principle and the Invariance of Areas

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In the plane, three distinct lines in general position (in which no lines are parallel and not all three lines intersect at a point) characterize a triangle. For simplicity, if the three lines are labeled L_1, L_2 and L_3, denote the intersection of lines L_i and L_j by $L_i \cdot L_j = L_j \cdot L_i$, if $i \neq j$, $1 \leq i, j \leq 3$.

What if we consider three pairs of parallel lines in general position? If the pairs are $\{L_i, L_i'\}$ for $i = 1, 2, 3$, then define $A_i = L_i \cdot L_{i+1}'$ and $B_i = L_i' \cdot L_{i+1}$ for indices taken modulo 3 (so that $L_4 = L_1$, etc.). How do you think the area of triangle $A_1A_2A_3$ relates to the area of triangle $B_1B_2B_3$? Draw some parallel lines, plot the corresponding triangles, and take a look. Any idea about the relationship between the two triangles? It turns out that the two areas are equal! This was proven by Liu [1]. He conjectured that a generalization of this theorem is true for any n sets of parallel lines for $n > 3$. His experiments with Geometer's Sketchpad appear to confirm the cases for n with $4 \leq n \leq 7$. Figure 1 includes a possible configuration for the case $n = 4$. Does it look plausible to you that the two quadrilaterals in Figure 1 have the same area?

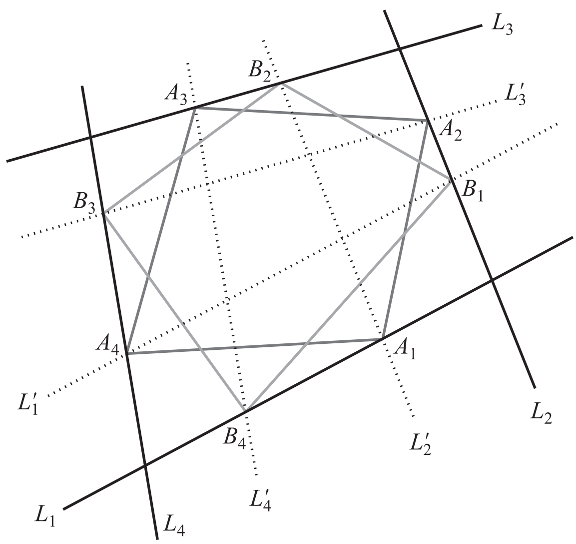

Figure 1 A possible configuration for the case $n = 4$.

In this article, we give two simple algebraic proofs of Liu's conjecture for any $n \geq 3$, under the assumption that the corresponding points of intersection of the different sets of parallel lines form well-positioned n-gons. After the two proofs, one of which involves exterior algebras, we explain why these proofs follow from an observation of Euclid. We start by stating the following proposition, which resolves Liu's conjecture.

Math. Mag. **91** (2018) 373–379. doi:10.1080/0025570X.2018.1520533
MSC: Primary 51M04
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/umma.

Proposition. *For $n \geq 3$, let $S_i = \{L_i, L_i'\}$, $1 \leq i \leq n$, be n (ordered) sets of parallel lines, such that lines from different sets intersect at a unique point. For $1 \leq i \leq n$, let A_i be the intersection of line L_i with L_{i+1}' and B_i be the intersection of the line L_i' with L_{i+1}, where $L_{n+1} = L_1$ and $L_{n+1}' = L_1'$. Then the area of the polygon $A_1 A_2 \ldots A_n$ equals that of the polygon $B_1 B_2 \ldots B_n$.*

Liu's proof for the case $n = 3$ is straightforward: after drawing a picture, he used the fact that parallel lines have the same slope and applied the determinant expression for the area of a triangle (equation 1). He also handled the degenerate case, namely that the three points A_1, A_2 and A_3 are collinear if and only if B_1, B_2 and B_3 are collinear. Since the vanishing of the 3×3 determinant characterizes collinearity, the same proof works for both cases. In what follows, we avoid this complication by assuming that our polygons are well-positioned.

Preliminaries

For the first proof, like Liu, we are going to use the determinant. For vectors in $\mathbb{R}^2$, an intuitive way of understanding a determinant is to regard it as the signed area of the associated parallelogram spanned by two vectors. For higher dimensions, the determinant may be viewed as the signed volume of the associated parallelepiped spanned by n vectors in $\mathbb{R}^n$. We will need two lemmas to proceed; the proofs are sketched below.

It is well known that if (x_i, y_i), $1 \leq i \leq 3$ are three points of a triangle in the plane arranged in a counterclockwise order, then the area of the triangle is given by

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}. \tag{1}$$

Using cofactor expansion along the last column, this determinant equals

$$\begin{aligned} &\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} \\ &\qquad = \frac{1}{2}[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3)], \end{aligned}$$

which we write symbolically as

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}.$$

Either by induction or by the application of Green's theorem, we get our first lemma.

Lemma 1. *For $n \geq 3$, let $P_1, P_2, \ldots, P_n$ be vertices (with coordinates (x_i, y_i), $1 \leq i \leq n$) in the plane that bound a (not necessarily convex) polygon, where $P_1, \ldots, P_n$ are oriented counterclockwise. The area of the polygon is given by*

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_1 & y_1 \end{vmatrix} = \frac{1}{2}\sum_{i=1}^{n}(x_i y_{i+1} - x_{i+1} y_i),$$

where we let $x_{n+1} = x_1$ and $y_{n+1} = y_1$.

For our second proof, we need some basic properties of an exterior algebra, or more specifically the wedge product $\wedge$. So let us recall that for the vector space $V = \mathbb{R}^2$ with the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, the exterior algebra $\bigwedge(V)$ is an associative linear algebra over $\mathbb{R}$ with basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_2\}$, subject to the relations $\mathbf{e}_1 \wedge \mathbf{e}_1 = 0$, $\mathbf{e}_2 \wedge \mathbf{e}_2 = 0$, $\mathbf{e}_1 \wedge \mathbf{e}_2 = -\mathbf{e}_2 \wedge \mathbf{e}_1$ and the condition that the span of 1, i.e., $\mathbb{R} \cdot 1$, lies in the center (i.e., commutes with other elements in the algebra). We refer to [2] for a formal definition and additional details about wedge products and exterior algebras. From the above relations, it is straightforward to derive a relationship between wedge products and determinants, as given in Lemma 2. In particular, for $\mathbf{v}_1, \mathbf{v}_2 \in V$, we have $\mathbf{v}_1 \wedge \mathbf{v}_2 = 0$ if and only if $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent, i.e., one is a multiple of the other.

Lemma 2. *Let* $\mathbf{v}_1 = x_1\mathbf{e}_1 + y_1\mathbf{e}_2$ *and* $\mathbf{v}_2 = x_2\mathbf{e}_1 + y_2\mathbf{e}_2$. *Then*

$$\mathbf{v}_1 \wedge \mathbf{v}_2 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \mathbf{e}_1 \wedge \mathbf{e}_2.$$

First proof: coordinate geometry approach

By suitable translation, we can guarantee that no line passes through the origin. Normalizing the constant term of the equations that describe the lines, we may assume that, for $1 \le i \le n$, L_i (resp. for L_i') is given by the equation $a_i x + b_i y = 1$ (resp. $k_i a_i x + k_i b_i y = 1$, where $k_i \neq 0$).

Denoting the coordinate of A_i (resp. B_i) by (x_i, y_i) (resp. (x_i', y_i')), we have by Cramer's Rule that, for $1 \le i \le n$,

$$x_i = \frac{k_{i+1}b_{i+1} - b_i}{k_{i+1}\begin{vmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{vmatrix}}, \; y_i = \frac{a_i - k_{i+1}a_{i+1}}{k_{i+1}\begin{vmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{vmatrix}} \tag{2}$$

and

$$x_i' = \frac{b_{i+1} - k_i b_i}{k_i\begin{vmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{vmatrix}}, \; y_i' = \frac{k_i a_i - a_{i+1}}{k_i\begin{vmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{vmatrix}}, \tag{3}$$

where we set $a_{n+1} = a_1$, $b_{n+1} = b_1$ and $k_{n+1} = k_1$.

Now by Lemma 1, the areas of $A_1 A_2 \dots A_n$ and $B_1 B_2 \dots B_n$ are given by

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_1 & y_1 \end{vmatrix} = \frac{1}{2}\sum_{i=1}^{n}(x_i y_{i+1} - x_{i+1} y_i) \tag{4}$$

and

$$\frac{1}{2}\begin{vmatrix} x_1' & y_1' \\ x_2' & y_2' \\ \vdots & \vdots \\ x_1' & y_1' \end{vmatrix} = \frac{1}{2}\sum_{i=1}^{n}(x_i' y_{i+1}' - x_{i+1}' y_i'). \tag{5}$$

Note that here our indices are considered modulo n, thus for example $a_{n+2} = a_2$. Then, substituting equation 2 into equation 4, the area of the n-gon $A_1 A_2 \dots A_n$ equals

$\frac{1}{2}$ times the summation from 1 to n of

$$\frac{1}{\begin{vmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{vmatrix}} - \frac{\begin{vmatrix} a_i & b_i \\ a_{i+2} & b_{i+2} \end{vmatrix}}{k_{i+1}\begin{vmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{vmatrix}\begin{vmatrix} a_{i+1} & b_{i+1} \\ a_{i+2} & b_{i+2} \end{vmatrix}} + \frac{1}{k_{i+1}k_{i+2}\begin{vmatrix} a_{i+1} & b_{i+1} \\ a_{i+2} & b_{i+2} \end{vmatrix}}.$$

Similarly, substituting equation 3 into equation 5, we get that the area of the n-gon $B_1B_2\ldots B_n$ equals $\frac{1}{2}$ times the summation from 1 to n of

$$\frac{1}{\begin{vmatrix} a_{i+1} & b_{i+1} \\ a_{i+2} & b_{i+2} \end{vmatrix}} - \frac{\begin{vmatrix} a_i & b_i \\ a_{i+2} & b_{i+2} \end{vmatrix}}{k_{i+1}\begin{vmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{vmatrix}\begin{vmatrix} a_{i+1} & b_{i+1} \\ a_{i+2} & b_{i+2} \end{vmatrix}} + \frac{1}{k_i k_{i+1}\begin{vmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{vmatrix}}.$$

In the last two formulas, the middle term in the summands is the same, while the first and the third terms are like terms in the summation under cyclic permutation. Shifting the index by 1, it is easy to see that the corresponding first and third sums are equal, so the area of $A_1A_2\ldots A_n$ equals that of $B_1B_2\ldots B_n$.

Second proof: exterior algebra approach

For notational convenience, let $\mathbf{v}_i := \overrightarrow{OA_i}$ be the position vector associated with the point A_i, $1 \le i \le n+1$, where $\mathbf{v}_1 = \mathbf{v}_{n+1}$. Similarly, let $\mathbf{v}'_i = \overrightarrow{OB_i}$, $1 \le i \le n+1$, where $\mathbf{v}'_1 = \mathbf{v}'_{n+1}$.

Lemma 3. *Using the above notation, it follows that*

$$\overrightarrow{A_iB_{i+1}} \wedge \overrightarrow{A_{i+1}B_i} = 0,$$

or, equivalently, that

$$\mathbf{v}_i \wedge \mathbf{v}_{i+1} - \mathbf{v}'_i \wedge \mathbf{v}'_{i+1} = \mathbf{v}_i \wedge \mathbf{v}'_i - \mathbf{v}_{i+1} \wedge \mathbf{v}'_{i+1}.$$

Proof. As defined in the proposition, notationally we have (see Figure 2)

$$\{A_i\} = L_i \cap L'_{i+1}, \{B_i\} = L'_i \cap L_{i+1}, \{A_{i+1}\} = L_{i+1} \cap L'_{i+2}, \text{ and}$$
$$\{B_{i+1}\} = L'_{i+1} \cap L_{i+2},$$

so that

$$A_i, B_{i+1} \in L'_{i+1} \text{ and } A_{i+1}, B_i \in L_{i+1}.$$

Since L'_{i+1} is parallel to L_{i+1}, it is clear that

$$\overrightarrow{A_iB_{i+1}} \wedge \overrightarrow{A_{i+1}B_i} = 0.$$

Because $\overrightarrow{A_iB_{i+1}} = \overrightarrow{OB_{i+1}} - \overrightarrow{OA_i} = \mathbf{v}'_{i+1} - \mathbf{v}_i$ and $\overrightarrow{A_{i+1}B_i} = \mathbf{v}'_i - \mathbf{v}_{i+1}$, then expanding linearly and rewriting finishes our proof. ∎

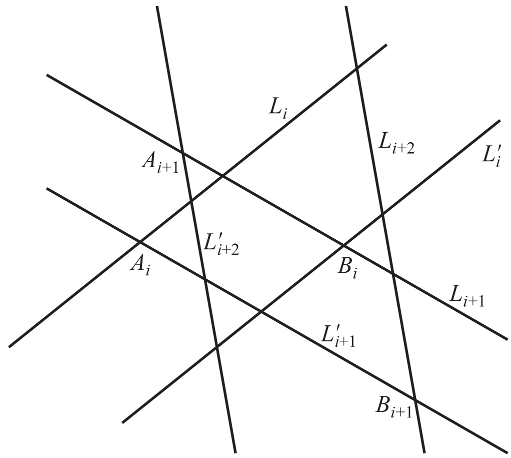

Figure 2 A simple but crucial observation.

Now to prove that the area of $A_1 \ldots A_n$ equals that of $B_1 \ldots B_n$, it suffices by Lemma 1 to show that

$$\frac{1}{2}\sum \begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1} \end{vmatrix} = \frac{1}{2}\sum \begin{vmatrix} x'_i & y'_i \\ x'_{i+1} & y'_{i+1} \end{vmatrix},$$

which by Lemma 2 is equivalent to

$$\frac{1}{2}\sum \mathbf{v}_i \wedge \mathbf{v}_{i+1} = \frac{1}{2}\sum \mathbf{v}'_i \wedge \mathbf{v}'_{i+1} \Leftrightarrow \sum(\mathbf{v}_i \wedge \mathbf{v}_{i+1} - \mathbf{v}'_i \wedge \mathbf{v}'_{i+1}) = 0 \tag{6}$$

Now Lemma 3 implies that

$$\sum(\mathbf{v}_i \wedge \mathbf{v}_{i+1} - \mathbf{v}'_i \wedge \mathbf{v}'_{i+1}) = \sum(\mathbf{v}_i \wedge \mathbf{v}'_i - \mathbf{v}_{i+1} \wedge \mathbf{v}'_{i+1}). \tag{7}$$

Also, the right-hand side of equation 7 equals 0, because

$$\begin{aligned}&\sum_{i=1}^{n}(\mathbf{v}_i \wedge \mathbf{v}'_i - \mathbf{v}_{i+1} \wedge \mathbf{v}'_{i+1})\\ &= (\mathbf{v}_1 \wedge \mathbf{v}'_1 - \mathbf{v}_2 \wedge \mathbf{v}'_2) + (\mathbf{v}_2 \wedge \mathbf{v}'_2 - \mathbf{v}_3 \wedge \mathbf{v}'_3) + \cdots + (\mathbf{v}_n \wedge \mathbf{v}'_n - \mathbf{v}_{n+1} \wedge \mathbf{v}'_{n+1})\\ &= \mathbf{v}_1 \wedge \mathbf{v}'_1 - \mathbf{v}_{n+1} \wedge \mathbf{v}'_{n+1} = 0.\end{aligned}$$

As a result, equation 6 holds and the proposition is proven.

Euclid's principle

In a nutshell, what we mean by Euclid's principle is that two triangles with the same base and the same height have the same area. To illustrate this, define two parallel lines L_1 and L_2, and fix two points B, C on L_1, and let A_1 and A_2 be two different points on L_2. Then the triangles A_1BC and A_2BC have the same area. This principle is encoded in the definition of the determinant and of an exterior algebra. For example, by Lemma 2, we can identify the area of triangle A_1BC with $\frac{1}{2}\overrightarrow{BC} \wedge \overrightarrow{BA_1}$, and the area of triangle A_2BC with $\frac{1}{2}\overrightarrow{BC} \wedge \overrightarrow{BA_2}$. Then,

$$\overrightarrow{BC} \wedge \overrightarrow{BA_2} = \overrightarrow{BC} \wedge (\overrightarrow{BA_1} + \overrightarrow{A_1A_2}) = \overrightarrow{BC} \wedge \overrightarrow{BA_1}$$

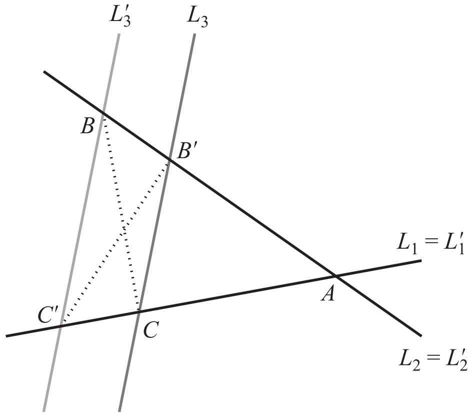

Figure 3 Deformation of parallel lines: the simplest case.

shows the equality of the two areas. Note that by Lemma 2 again, the wedge product is invariant under linear transformations of determinant 1; hence, it is invariant under rotation (we caution the readers that reflection causes the area to change sign).

Now it becomes a simple exercise to interpret the proof of the Pythagorean theorem in Euclid's Elements in terms of an exterior algebra. So, in a sense, our proofs are based on Euclid's principle. How does this principle directly contribute to a proof of the above proposition? We will sketch a proof and leave the details to the interested readers. For this purpose, we will use the notion of the deformation of parallel lines, starting with the case when the pair of parallel lines L_i and L_i' overlap, i.e., when $L_i = L_i'$, for each i. In this case, the result of the proposition is trivial. Next, let's look at the situation when $n = 3$, $L_1 = L_1'$ and $L_2 = L_2'$, but $L_3 \neq L_3'$.

Then denoting intersection points as in Figure 3, we see that the triangles ABC and $AB'C'$ overlap in triangle $AB'C$. Hence it suffices to compare the area of the triangle BCB' and the area of the triangle $C'CB'$. These areas are equal by Euclid's principle. Using Lemma 2 and Euclid's principle, the general case follows from direct verification that deforming one line from a fixed pair of parallel lines (while leaving other pairs of parallel lines unchanged) will modify the polygons locally and the local changes of areas are equal.

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Summary. Previously, B. Liu constructed two n-gons from the intersection of pairs of parallel lines. He conjectured that the areas of two n-gons were equivalent. In this article, we provide two proofs of the conjecture and relate the proofs to a principle of Euclid's. The first proof is based on coordinate geometry and applies determinants. The second proof applies the structure of an exterior algebra.

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P	V	S	N	P		F	O	I	L		C	L	E	W
D	I	C	E	R		J	U	N	E		H	I	R	E
E	D	R	A	Y	G	O	I	N	S		A	N	D	A
S	I	U		I	R	R			S	A	I	G	O	N
		B	E	N	E	D	I	C	T	G	R	O	S	S
			E	G	G		C	O	H	N	S			
P	O	L	L			D	A	N	A	E		M	A	A
T	H	I	S	T	L	E		U	N	S	N	A	R	L
A	M	S		R	P	L	U	S			A	X	I	S
			T	E	S	T	A		A	B	C			
K	A	R	E	N	P	A	R	S	H	A	L	L		
A	G	E	N	D	A			C	O	T		O	D	B
B	A	T	S		C	A	T	H	Y	O	N	E	I	L
U	P	T	O		E	C	R	U		N	A	I	V	E
L	E	A	R		S	E	E	R		S	Y	L	O	W

The Pythagorean Theorem and the Angle Sum and Difference Identitites

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The trigonometric identities

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

and

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

each imply the Pythagorean identity and consequently the Pythagorean theorem by setting $\alpha+\beta=\pi/2$ and $\alpha-\beta=0$, respectively. Although these identities are typically derived using the Pythagorean theorem, that is not necessary (see [3, p. 46], [5], [6], and [8]), and circular arguments can be avoided.

Here we show how the sum and difference formulas for the tangent function,

$$\tan(\alpha\pm\beta) = \frac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)},$$

can be used to prove the Pythagorean theorem.

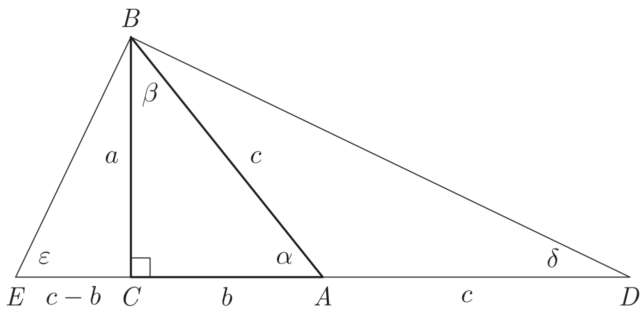

Figure 1 Extending one side of a right triangle in two directions.

As shown in Figure 1, we extend the leg CA of the right triangle BCA beyond A by c to get D. We have

$$\frac{b+c}{a} = \tan(\beta+\delta) = \frac{\tan(\beta)+\tan(\delta)}{1-\tan(\beta)\tan(\delta)} = \frac{\frac{b}{a}+\frac{a}{b+c}}{1-\frac{b}{a}\cdot\frac{a}{b+c}} = \frac{b^2+bc+a^2}{ac},$$

and $c^2=a^2+b^2$ follows.

Math. Mag. **91** (2018) 380–381. doi:10.1080/0025570X.2018.1467670
MSC: Primary 51M04, Secondary 00A05; 00A08

A similar argument using the angle difference formula works when AC is extended beyond C, giving

$$\frac{c-b}{a} = \tan(\varepsilon - \beta) = \frac{\tan(\varepsilon) - \tan(\beta)}{1 + \tan(\varepsilon)\tan(\beta)} = \frac{\frac{a}{c-b} - \frac{b}{a}}{1 + \frac{a}{c-b} \cdot \frac{b}{a}} = \frac{a^2 - bc + b^2}{ac},$$

and $c^2 = a^2 + b^2$ follows again.

This same diagram was used by Gottfried Wilhelm Leibniz to give a geometric proof of the Pythagorean theorem (see [**2**, Proof 53], [**1**], and [**7**]). The triangle DBE is a right triangle with circumcircle centered at A. By the similarity of triangles BCD and BCE, or by the power-of-a-point theorem, we have $a^2 = (c-b)(c+b)$, and thus the Pythagorean theorem.

Note that our proof is independent of the sum and difference formulas for sine and cosine since the sum and difference formulas for tangent have direct geometric proofs that—unlike the typical derivations in most textbooks—do not rely on the corresponding formulas for sine or cosine; see [**3**, p. 47] and [**4**].

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Summary. The sum and difference identities for tangent are used to prove the Pythagorean theorem.

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Proving Euler's Four-Square Lemma using Linear Algebra

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It is well known that Lagrange showed in 1770 that every positive integer can be written as the sum of at most four perfect squares. An essential element in the classical proof of this theorem is Euler's lemma: if m, n are positive integers which can each be written as the sum of four perfect squares, then the same is true of their product. Specifically, the lemma states that

$$\begin{aligned}(a^2+b^2+c^2+d^2)(p^2+q^2+r^2+s^2)\\ &= (ap+bq+cr+ds)^2+(-aq+bp-cs+dr)^2\\ &\quad + (-ar+bs+cp-dq)^2+(-as-br+cq+dp)^2.\end{aligned}$$

This lemma then allows the general theorem to be reduced to the case where the integer is an odd prime.

The proof of the lemma is normally accomplished by a brute force expansion of the right side and the observation that all cross product terms cancel (characterized as "not suitable for the printed page" in [1]) or using the properties of norms on the quaternions. (The "brute force" approach does allow the easy observation that the lemma is valid in any commutative ring.) It is the purpose of this note to observe that Euler's lemma is in fact a corollary of the following two well-known theorems in linear algebra.

(I) Any orthogonal set of nonzero vectors in $\mathbb{R}^n$ is linearly independent, and so any set of n such vectors is a basis for $\mathbb{R}^n$.

(II) If $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is an orthogonal basis of $\mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$, then

$$\mathbf{v} = \sum_{i=1}^{n} \frac{\mathbf{v}\cdot\mathbf{w}_i}{||\ \mathbf{w}_i\ ||^2}.$$

Among many other possibilities, the reader may consult pp. 279–280 of [2].

Let $\mathbf{w}_1 = (p, q, r, s)$ be a nonzero vector in $\mathbb{R}^4$. It is then routine to verify that $\mathbf{B} = \{\mathbf{w}_1, \mathbf{w}_2 = (-q, p, -s, r), \mathbf{w}_3 = (-r, s, p, -q), \mathbf{w}_4 = (-s, -r, q, p)\}$ is an orthogonal set and so a basis for $\mathbb{R}^4$, and that $\|\mathbf{w}_i\|^2 = p^2+q^2+r^2+s^2$, for $1 \le i \le 4$. Let K represent this expression, and let $\mathbf{v} = (a, b, c, d)$.

By (II) above,

$$\mathbf{v} = \sum_{i=1}^{4} \frac{\mathbf{v}\cdot\mathbf{w}_i}{K}\mathbf{w}_i.$$

Multiplying both sides by K and forming the dot product of $K\mathbf{v}$ with itself, it follows that $K^2(\mathbf{v}\cdot\mathbf{v})$ equals the dot product of $\sum_{i=1}^{4}(\mathbf{v}\cdot\mathbf{w}_i)\mathbf{w}_i$ with itself. By the orthogonality

Math. Mag. **91** (2018) 384–385. doi:10.1080/0025570X.2018.1465745
MSC: Primary 11D25, Secondary 11D85; 15A03

of **B**, this dot product reduces to $K\sum_{i=1}^{4}(\mathbf{v}\cdot\mathbf{w}_i)^2$. Cancelling K from both sides then yields the identity of Euler's lemma.

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Summary. Euler's Four-Square Lemma is proved using two well-known results from linear algebra.

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Letter to the Editor

I noticed the article, "A Proof of the Law of Sines Using the Law of Cosines" [1] in the June 2017 issue. Here is an alternative proof.

$$a^2 = b^2 + c^2 - 2bc\cos A = b^2 + c^2 + 2bc\cos(B + C)$$
$$= \big(b\cos C + c\cos B\big)^2 + \big(b\sin C - c\sin B\big)^2.$$

Since $b\cos C + c\cos B = a$, then $b\sin C = c\sin B$.

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–Submitted by Li Zhou, Polk State College, Winter Haven, FL 33881
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Partiti Puzzle Solution

15 159	11 47	16 169	13 247	15 159	11 47
9 36	10 28	8 35	8 8	9 36	10 28
18 459	8 17	13 49	8 17	13 49	13 157
6 6	11 38	8 26	16 358	8 26	11 38
6 15	13 49	1 1	13 49	8 17	13 49
18 378	6 6	23 3578	8 26	11 38	13 256

ACROSS

1. One of the Millennium Prize Problems from the Clay Mathematics Inst.
6. Student acronym for multiplying two binomials
10. Lower corner of a sail (and a homophone of what Sherlock Holmes might seek)
14. Someone at the craps table?
15. Mathematician Huh who (with collaborators) proved the Heron-Rota-Welsh Conjecture
16. Acquire new faculty
17. * Speaker from Pomona College whose address is on the past 50 years of African Americans in mathematics
19. Live music lead-in: "One ___ two . . ."
20. Higher ed. sch. in Carbondale
21. Like $\sqrt{2}$ and π and e: Abbr.
22. Tony-nominated musical: "Miss ___"
24. * Speaker from UC San Diego who will give three invited colloquium lectures on complex multiplication
28. It may be scrambled
29. Attorney Roy, who worked for McCarthy and Trump, and mathematician Paul, known for "Free rings and their relations"
30. Survey
33. Tai-___ Bradley, recent host of PBS' "Infinite Series" YouTube channel
34. * One managing org. of the JMM
37. Prickly flowering plant, a symbol of Scotland
40. Turn a tangle into the unknot, perhaps?
42. * One managing org. of the JMM
43. Symbol for the positive real numbers
45. The Cartesian plane has an x and a y one
46. Head, to mathematicians Peano and Fubini
47. Mathematician Mochizuki has claimed to have a proof of this conjecture
49. * Speaker from University of Virginia whose address is on American mathematics in the 1920's
54. You may set one each day at the conference, to stay organized and punctual
55. cos / sin
56. Wu-Tang Clan founding member: Abbr.
59. They're kept in the dugout when not in use
60. * Speaker whose address is entitled "Big data, inequality, and democracy"
64. "What are you ___?"
65. Light beige color
66. Text by Paul Halmos: "___ Set Theory"
67. Shakespearean King
68. Fortune teller
69. Norwegian mathematician, eponym of theorems about subgroups

DOWN

1. Laplace's and Poisson's equations, e.g.
2. "Veni, ___, vici"
3. According to TLC, it's a "guy that can't get no love from me"
4. US gov. agency that awards grants for projects in music, dance or theater
5. Inquiring a bit too closely into one's personal affairs
6. Narrow inlet between cliffs
7. Yes, to mathematicians Galois and Villani
8. Holiday ___
9. $<$
10. Department heads
11. Field-specific jargon
12. Mathematician to whom everyone measures their collaboration distance
13. Gets an infant eating solid food instead of breast milk
18. Golfer ___ Norman, a.k.a The Shark
23. Actress Moorehead from "Bewitched" and "Citizen Kane"
25. Wriggly fish
26. Acronym for museum (in Boston, Philadelphia, and LA) featuring recent paintings, sculptures, etc.
27. Military shorthand for the 48 states excluding Hawaii and Alaska
30. Sch. group for moms and dads
31. Unit of resistance symbolized by Ω
32. Fleur-de-___
33. Greek letter used to symbolize change
34. Math.___(1,6,4,3) will return 6 in JavaScript
35. The Diamondbacks, on scoreboards: Abbr.
36. Politicians Franken and Gore
38. You might make a scatterplot to look for this
39. Some objects of study in functional analysis
41. Table salt, to a chemist
44. Egypt and Syria, once: Abbr.
46. Generalization of a matrix that is common in mathematical physics
47. Sailor's call
48. Conductors' sticks
49. Capital of Afghanistan
50. Open-mouthed in surprise
51. One-named actress from "Parks and Recreation" and "Good Girls"
52. Mathematician Issai with many things named for him, including a polynomial, a product, and an inequality
53. Trompe ___
57. Male opera star
58. Made bubbles
61. It's always worth one in cribbage
62. Uno + due
63. Senate vote

Math. Mag. **91** (2018) 386–387. doi:10.1080/0025570X.2018.1529519 © Mathematical Association of America

Joint Mathematics Meetings 2019

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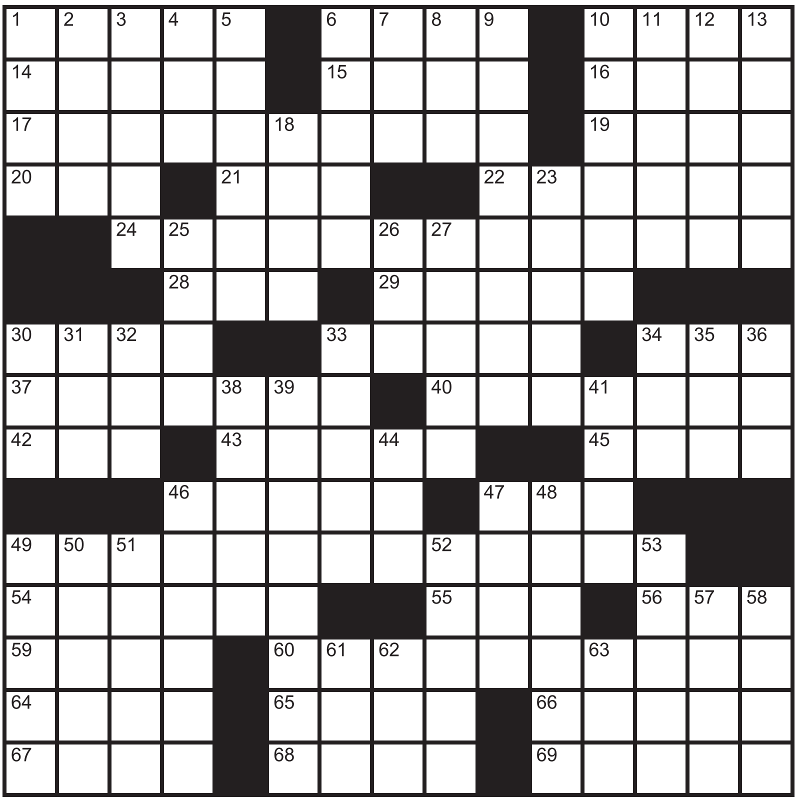

Clues start at left, on page 386. The Solution is on page 379.

This crossword puzzle and solution are open access and available for download on Taylor & Francis' website for the MAGAZINE (https://maa.tandfonline.com).

PROBLEMS

Proposals

To be considered for publication, solutions should be received by May 1, 2019.

2056. *Proposed by Armend Sh. Shabani, University of Prishtina, Republic of Kosovo.*

Let n and k be positive integers. Regard the $2k$ numbers $1, 2, 3, \ldots, 2k$ as letters of an alphabet. Find the number of length-n words (using letters from said alphabet with repetitions allowed) in which no two odd letters are adjacent.

2057. *Proposed by Enrique Treviño, Lake Forest College, Lake Forest, IL.*

Let the sequence $x_1, x_2, \ldots, x_n, \ldots$ be defined by $x_1 = 0$, $x_2 = x_3 = x_4 = x_5 = 1$, $x_6 = x_7 = \cdots = x_{23} = 0$, $x_{24} = x_{25} = \cdots = x_{119} = 1, \ldots, x_{(2k-1)!} = \cdots = x_{(2k)!-1} = 0$, $x_{(2k)!} = \cdots = x_{(2k+1)!-1} = 1, \ldots$ (for $k = 1, 2, 3, \ldots$).

(i) For $n \geq 1$ let

$$a_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

be the nth arithmetic average of (x_n). Is the sequence $a_1, a_2, \ldots, a_n, \ldots$ convergent? If so, find its limit.

(ii) For $n \geq 1$ let

$$b_n = \frac{1}{H_i}\sum_{i=1}^{n} \frac{x_i}{i}$$

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Proposals and solutions should be written in a style appropriate for this MAGAZINE.

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be the n-th harmonically weighted average of (x_n), where

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

is the n-th harmonic sum. Is the sequence $b_1, b_2, \ldots, b_n, \ldots$ convergent? If so, find its limit.

2058. *Proposed by Gregory Dresden, Saimon Islam (student) and Jiahao Zhang (student), Washington & Lee University, Lexington, VA.*

Let a be a rational number such that the polynomial

$$f(x) = x^6 + 3x^5 - ax^4 - (2a+5)x^3 - ax^2 + 3x + 1$$

is irreducible over $\mathbb{Q}$, and let F be the splitting field for $f(x)$ over $\mathbb{Q}$. Find the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ (up to isomorphism).

2059. *Proposed by Andrew Wu, St. Albans School, McLean, VA.*

Let triangle $\triangle ABC$ be acute and scalene with orthocenter H, altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, and circumcircle Ω. Let Γ be the circle with diameter $\overline{AH}$. Circles Γ and Ω intersect at A and at a second point K. Let point P lie on Γ so that $\overline{HP}$ is parallel to $\overline{EF}$. Let M be the midpoint of $\overline{BC}$. Let $\overleftrightarrow{AM}$ intersect Ω at $R \neq A$, and $\overline{EF}$ at Q. Let $\overleftrightarrow{PQ}$ meet Γ again at $X \neq P$. Show that $\overline{DX}$ and $\overline{KR}$ concur on Γ.

2060. *Proposed by Su Pernu Mero, Valenciana GTO, Mexico.*

Let A, B, C be independent standard normal random variables, i.e., the PDF of each is $f(x) = \exp(-x^2/2)/\sqrt{2\pi}$. Consider the random matrix

$$M = \begin{pmatrix} A & B/\sqrt{2} \\ B/\sqrt{2} & C \end{pmatrix}.$$

What is the probability that M is positive definite?

Quickies

1085. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.*

Let a, b, c be real numbers in the interval $(0, 1)$. Prove that

$$\frac{1}{1-a^3} + \frac{1}{1-b^3} + \frac{1}{1-c^3} + \frac{3}{1-abc} > \frac{6}{1-a^2b^2c^2}.$$

1086. *Proposed by Warut Suksompong (student), Stanford University, Stanford, CA.*

Find the largest natural number n with the property that there exist n points on the plane, not all collinear, such that no three noncollinear points are vertices of an obtuse triangle.

Solutions

A Pythagorean relation in regular (2n+1)-gons **December 2017**

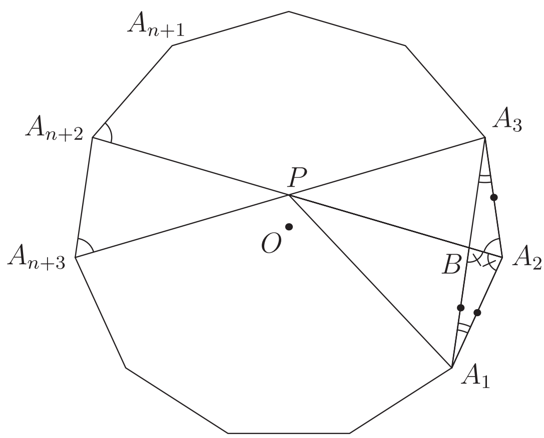
2031. *Proposed by Barış Burçin Demir, Ankara, Turkey.*

Let n be an integer, $n \geq 2$. Let $A_1A_2A_3 \ldots A_{2n+1}$ be a regular polygon with $2n+1$ sides. Let P be the intersection of the segments $\overline{A_2A_{n+2}}$ and $\overline{A_3A_{n+3}}$. Prove that

$$(A_1P)^2 = (A_2A_3)^2 + (A_3P)^2.$$

Solution by Skidmore College Problem Group, Saratoga Springs, New York.

Denote the polygon $A_1A_2 \ldots A_{2n+1}$ by $\mathcal{F}$ and let $\alpha = \pi/(2n+1)$. The angles of $\mathcal{F}$ are all equal to $\angle A_1A_2A_3 = (2n-1)\alpha$. On the other hand, $\angle A_1A_3A_2 = \angle A_3A_1A_2 = (\pi - \angle A_1A_2A_3)/2 = [\pi - (2n-1)\alpha]/2 = \alpha$, since triangle $\triangle A_1A_2A_3$ is isosceles with $A_1A_2 = A_2A_3$.



Let $\beta = \angle A_2A_{n+2}A_{n+1} = \angle A_{n+2}A_2A_3$. The sum of angles of the $(n+1)$-gon $A_2A_3 \ldots A_{n+1}A_{n+2}$ is $(n-1)\pi = 2\beta + (n-1)\cdot(2n-1)\alpha$, hence $\beta = (n-1)[\pi - (2n-1)\alpha]/2 = (n-1)\alpha$. Thus, $\angle A_1A_2P = \angle A_1A_2A_3 - \angle PA_2A_3 = (2n-1)\alpha - (n-1)\alpha = n\alpha$. Let B be the intersection point of $\overline{PA_2}$ and $\overline{A_1A_3}$. Then $\angle A_1BA_2 = \pi - \angle A_3A_1A_2 - \angle A_1A_2P = (2n+1)\alpha - (n+1)\alpha = n\alpha = \angle A_1A_2P = \angle A_1A_2B$. Thus, in the isosceles triangle $\triangle A_1BA_2$ we have $A_2B = 2A_1A_2 \cos n\alpha$.

It is clear that the polygons $A_2A_3 \ldots A_{n+2}$ and $A_3A_4 \ldots A_{n+3}$ are congruent as they are related through a rotation of angle 2α from the center O of $\mathcal{F}$, hence $\angle A_2PA_3 = 2\alpha$, and so $\angle PA_3A_2 = \pi - \angle PA_2A_3 - \angle A_2PA_3 = \pi - (n-1)\alpha - 2\alpha = n\alpha$. Since $\angle A_3BP = \angle A_1BA_2 = n\alpha = \angle A_2A_3P$ and $\angle BA_3P = \angle A_2A_3P - \angle A_1A_3A_2 = n\alpha - \alpha = (n-1)\alpha = \angle PA_2A_3$, we have a similarity $\triangle A_3BP \sim \triangle A_2A_3P$, hence $A_2P \cdot BP = (A_3P)^2$. By the law of cosines in triangle $\triangle A_1A_2P$,

$$\begin{aligned}(A_1P)^2 &= (A_1A_2)^2 + (A_2P)^2 - 2A_2P \cdot A_1A_2 \cos n\alpha = (A_1A_2)^2 + (A_2P)^2 - A_2P \cdot A_2B \\ &= (A_1A_2)^2 + A_2P(A_2P - A_2B) = (A_1A_2)^2 + A_2P \cdot BP \\ &= (A_2A_3)^2 + (A_3P)^2.\end{aligned}$$

Also solved by Michel Bataille, Eric Compton, Carol H. Chandler, Timothy L. Chandler, Con Amore Problem Group (Denmark), Ivko Dimitrić, Habib Y. Far, Dmitry Fleischman, Marty Getz

& Dixon Jones, Eugene Herman, Nathan Kahl, Kee-Wai Lau, Qian Liu & Shamil Dzhatdoyev, Elias Lampakis (Greece), Jerry Minkus, José H. Nieto (Venezuela), Randy K. Schwartz, Volkhard Schindler (Germany), Michael Vowe, and the proposer.

The best L^1 approximation by a constant **December 2017**

2032. *Proposed by Noah H. Rhee, University of Missouri–Kansas City, MO.*

Let a, b be real numbers with $a < b$, and let f be a continuous, strictly increasing function on the closed interval $[a, b]$. For $y \in \mathbb{R}$, define

$$E(y) = \int_a^b |f(x) - y|\, dx.$$

Prove that $E(y)$ has a minimum value as y varies in $\mathbb{R}$, and find all y for which the minimum is attained.

Solution by Reiner Martin, Bad Soden am Taunus, Germany.
We prove that the minimum is attained at $y_0 = f((a + b)/2)$, and only there.

Let $x_0 = (a + b)/2$. Since $a < x_0 < b$ and f is strictly increasing, we have $f(a) < y_0 < f(b)$. Let $y_1 > y_0$. If $y_1 \le f(b)$, by the intermediate value theorem there exists $x_1 \in (x_0, b]$ such that $f(x_1) = y_1$. Then we have

$$\begin{aligned} E(y_1) - E(y_0) &= \int_a^{x_1} (y_1 - f(x))\, dx + \int_{x_1}^b (f(x) - y_1)\, dx \\ &\quad - \left[\int_a^{x_0} (y_0 - f(x))\, dx + \int_{x_0}^b (f(x) - y_0)\, dx\right] \\ &= \int_a^{x_0} (y_1 - y_0)\, dx + \int_{x_0}^{x_1} (y_0 + y_1 - 2f(x))\, dx + \int_{x_1}^b (y_0 - y_1)\, dx \\ &> \int_a^{x_0} (y_1 - y_0)\, dx + \int_{x_0}^{x_1} (y_0 - y_1)\, dx + \int_{x_1}^b (y_0 - y_1)\, dx \\ &= (2x_0 - a - b)(y_1 - y_0) = 0 \end{aligned}$$

since $f(x) < f(x_1) = y_1$ for $x \in [x_0, x_1)$ as f is strictly increasing. Thus, $E(y_1) > E(y_0)$ for $y_1 \in (y_0, f(b)]$. If $y_1 > f(b)$, then

$$E(y_1) = \int_a^b (y_1 - f(x))\, dx > \int_a^b (f(b) - f(x))\, dx = E(b) > E(y_0).$$

This shows that $E(y_1) > E(y_0)$ for $y_1 > y_0$. An analogous argument shows that $E(y_1) < E(y_0)$ for $y_1 < y_0$, concluding the proof.

Also solved by Ulrich Abel (Germany), Michel Bataille (France), Elton Bojaxhiu and Enkel Hysnelaj, Robert Calcaterra, Richard Daquila, Robert L. Doucette, Shamil Dzhatdoyev and Qian Liu (students), Dmitry Fleischman, Eugene A. Herman, Elias Lampakis (Greece), Missouri State University Problem Solving Group, Northwestern University Math Problem Solving Group, Moubinool Omarjee (France), John Alexis Osorio Monsalve (Colombia), Edward Schmeichel, and the proposer.

An application of Hall's marriage theorem **December 2017**

2033. *Proposed by Yoshihiro Tanaka, Hokkaido University, Sapporo, Japan.*

A deck is the collection of all 52 pairs ("cards") of the form (n, s) where $1 \leq n \leq 13$ is the number on the card, and the suit s of the card is one of the symbols $\diamondsuit$, $\heartsuit$, $\spadesuit$, $\clubsuit$. Given an arbitrary partition of a deck into 13 sets $S_1, S_2, \ldots, S_{13}$ of 4 cards each, prove that there exists a corresponding partition C_1, C_2, C_3, C_4 of the deck into 4 sets of 13 cards each, such that each of the parts C_i $(1 \leq i \leq 4)$ satisfies:

(i) C_i has one card from S_j for $1 \leq j \leq 13$, and

(ii) the cards in C_i all have different numbers.

Solution by Edward Schmeichel, San José State University, San José, CA.
For given integers $\nu, \sigma \geq 1$ we consider a more general deck of $\nu\sigma$ cards, each a pair (n, s) where n $(1 \leq n \leq \nu)$ is the card's number, and s $(1 \leq s \leq \sigma)$ is the card's suit.

Consider a partition of such deck into ν sets $S_1, \ldots, S_\nu$ of σ cards each. For $1 \leq k \leq \nu$, the union $T = S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_k}$ of any k of the parts must contain cards of at least k distinct numbers, since T has $k\sigma$ elements and each number occurs on only σ cards. By Hall's marriage theorem (Theorem 5.1.1 in Marshall Hall, *Combinatorial Theory.* John Wiley & Sons, Inc., Hoboken, NJ, 1999, http://dx.doi.org/10.1002/9781118032862 or Wikipedia's article https://en.wikipedia.org/wiki/Hall's_marriage_theorem), there exists a set C_1 of ν different-numbered cards containing exactly one card from each S_i. If all ν cards from C_1 are removed from the subsets S_i, we are left with ν subsets $S'_1, \ldots, S'_\nu$ of $\sigma - 1$ cards each. A further application of Hall's theorem yields a set C_2 of ν different-numbered cards having one card from each S'_i (*a fortiori*, one from each S_i). Removing the cards in C_2 from $S'_1, \ldots, S'_\nu$ and proceeding successively we obtain a partition of the entire deck into sets $C_1, C_2, \ldots, C_\sigma$ with each part C_j having ν different-numbered cards, one from each S_i. The particular case $\nu = 13$, $\sigma = 4$ of this construction solves the problem.

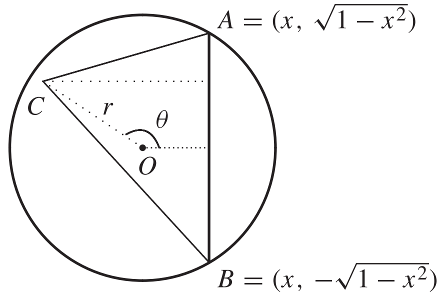
Also solved by Armstrong Problem Solvers, Elton Bojaxhiu (Germany) and Enkel Hysnelaj (Australia), Robert Calcaterra, Stephen C. Locke, José H. Nieto (Venezuela), Michael Reid, Dinesh Sarvates and William Cowden, Jacob Siehler, and the proposer. There was 1 incomplete or incorrect solution.

A geometric probability question in the circle **December 2017**

2034. *Proposed by Julien Sorel, Piatra Neamt, PNI, Romania.*

Let $\mathfrak{C}$ be a circle. Two points A, B are independently chosen on the circumference of $\mathfrak{C}$, uniformly at random. Two further points C, D are independently chosen in the interior of $\mathfrak{C}$ uniformly at random. What is the probability that D shall lie inside $\triangle ABC$?

Solution by Tyler Helton (student), Noah Rosenbalm (student) and Andy Simoson, King University, Bristol, Tennessee.
We may assume that $\mathfrak{C}$ is a unit circle centered at the origin $O(0, 0)$ and choose coordinate axes so that A and B are the points $(x, \pm\sqrt{1 - x^2})$ with $x \geq 0$. By rotational symmetry of the involved uniform distributions, this is no loss of generality. Since A, B are chosen uniformly and independently, the angle $\omega = \frac{1}{2}\angle AOB$ is uniformly distributed in $[0, \pi/2]$, while $x = \cos\omega$ has probability density function $f(x) = (2/\pi)(1 - x^2)^{-1/2}$ on the interval $[0, 1]$. By symmetry under reflection, conditioning C to lie in quadrants I or II does not change the probability. We have $C = (r\cos\theta, r\sin\theta)$ in terms of independent variables θ uniform in $[0, \pi]$ and r on $[0, 1]$, the latter with probability density function $g(r) = 2r$, by the formula for the area element in polar coordinates. Thus, the joint probability density function for



(x, r, θ) is $h(x, r, \theta) = (1/\pi) f(x) g(r) = (4r/\pi^2)(1-x^2)^{-1/2}$ on $[0, 1] \times [0, 1] \times [0, \pi]$. The area of $\triangle ABC$ is $[ABC] = |x - r\cos\theta|(1-x^2)^{1/2}$ (see the figure above). With D chosen uniformly at random, the probability of the event $\mathcal{E}$ that D lies within $\triangle ABC$ is

$$\mathbb{P}[\mathcal{E} \mid A, B, C] = \frac{[ABC]}{\pi} = \frac{1}{\pi}|r\cos\theta - x|(1-x^2)^{1/2}.$$

Thus, the desired probability is given by

$$\int_0^1 \int_0^1 \int_0^\pi \mathbb{P}[\mathcal{E} \mid A, B, C] \cdot h(x, r, \theta)\, d\theta\, dr\, dx = \int_0^1 \int_0^1 \int_0^\pi \frac{4r}{\pi^3}|r\cos\theta - x|\, d\theta\, dr\, dx$$
$$= \frac{5}{4\pi^2} \approx 12.7\%.$$

Also solved by Randy K. Schwartz, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Jan A. Grzesik, Robert Calcaterra, Peter McPolin (UK), and the proposer.

Algebraic numbers with common decimal tails **December 2017**

2035. *Proposed by Gregory Dresden, Prakriti Panthi (student), Anukriti Shrestha (student) and Jiahao Zhang (student), Washington & Lee University, Lexington, VA.*

Two real numbers x, y are said to *have a common decimal part* if $xy < 0$ and $x + y$ is an integer, or else $xy \geq 0$ and $x - y$ is an integer. More concretely, this means that the decimal expansions of x, y are of the forms

$$\pm a_m a_{m-1} \ldots a_1 a_0 \,.\, d_1 d_2 d_3 \ldots,$$
$$\pm b_n b_{n-1} \ldots b_1 b_0 \,.\, d_1 d_2 d_3 \ldots,$$

where the common decimal part is $0.d_1 d_2 d_3 \ldots$.

Find all polynomials of degree at least 2 with integer coefficients, all roots real, and irreducible over the rationals, whose roots have pairwise common decimal tails.

Solution by Jacob Siehler, Gustavus Adolphus College, Saint Peter, MN.

We show that the property in the statement of the problem characterizes those polynomials that are quadratic with integer coefficients and irreducible over $\mathbb{Q}$ such that $ac < 0$ and $a|b$. (The irreducibility amounts to the discriminant $b^2 - 4ac$ not being a perfect square.)

Lemma 1. A polynomial with integer coefficients, irreducible over $\mathbb{Q}$, cannot have two roots that differ by an integer.

Proof. Suppose f were such a polynomial, with two roots α and $\alpha + c$, where c is a nonzero integer. By irreducibility of f, the Galois group of the splitting field K of f over $\mathbb{Q}$ acts transitively on the roots of f, so there is a field automorphism ϕ of K with $\phi(\alpha) = \alpha + c$. Since c is rational and hence fixed by ϕ, it follows by induction that $\alpha, \alpha + c, \alpha + 2c, \ldots, \alpha + nc, \ldots$ are distinct roots of f (each successive one the image under ϕ of the previous), contradicting the fact that a polynomial has only finitely many roots. ■

Lemma 1 implies that a polynomial f meeting the requirements of the problem cannot have two roots of the same sign. Therefore, any such f must necessarily have exactly two distinct roots of opposite signs. Since f is irreducible over $\mathbb{Q}$ and non-linear, it has no repeated roots and no zero root. Thus, f must be quadratic and its two roots must be real of opposite signs with common decimal tails (i.e., integer sum). The following lemma characterizing these quadratics in terms of their coefficients concludes our solution.

Lemma 2. Let $f(x) = ax^2 + bx + c$ be irreducible over $\mathbb{Q}$, with integer coefficients (where $a \neq 0$). The roots of f are real of opposite sign and have common decimal tails if and only if $ac < 0$ and $a|b$.

Proof. Since f is quadratic and irreducible over $\mathbb{Q}$, it has exactly two distinct roots α and β.

If α and β are opposite-sign reals with common decimal tails, then $-b/a = \alpha + \beta$ is an integer, and furthermore $c/a = \alpha\beta < 0$, hence $ac < 0$ also.

Conversely, if $ac < 0$ and $a|b$, then $b^2 - 4ac > 0$, so α and β are real numbers; furthermore, they have opposite signs since $\alpha\beta = c/a < 0$ (as $ac < 0$ by assumption). By the assumption $a|b$, it follows that $\alpha + \beta = -b/a$ is an integer, so α and β have common decimal tails. ■

Also solved by Robert Calcaterra, Michael Reid, and the proposer.

Answers

Solutions to the Quickies from page 389.

A1085. For all positive x, y, z we have the well-known inequality

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2,$$

which is the special case $t = 1$ of Schur's inequality $x^t(x - y)(x - z) + y^t(y - z)(y - x) + z^t(z - x)(z - y) \geq 0$ valid for nonnegative x, y, z. (Wikipedia page: https://en.wikipedia.org/wiki/Schur's_inequality.) Taking $x = a^n$, $y = b^n$ and $z = c^n$ above for $n = 0, 1, 2, \ldots$, and adding the resulting inequalities using the geometric formula, we obtain

$$\begin{aligned}\frac{1}{1-a^3}+\frac{1}{1-b^3}+\frac{1}{1-c^3}+\frac{3}{abc} &= \sum_{n=0}^{\infty}\{a^{3n}+b^{3n}+c^{3n}+3(abc)^n\}\\ &\geq \sum_{n=0}^{\infty}\{(a^2b)^n+(a^2c)^n+(ab^2)^n+(b^2c)^n+(ac^2)^n+(bc^2)^n\}\\ &= \frac{1}{1-a^2b}+\frac{1}{1-a^2c}+\frac{1}{1-ab^2}+\frac{1}{1-b^2c}+\frac{1}{1-ac^2}+\frac{1}{1-bc^2}\\ &> \frac{6}{1-a^2b^2c^2}\end{aligned}$$

since, by the hypothesis that a, b, c are in (0, 1), we have that a^3, b^3, c^3, a^2b, a^2c, ab^2, b^2c, ac^2, bc^2, abc and $a^2b^2c^2$ all are in (0, 1), and moreover a^2b, a^2c, ab^2, b^2c, ac^2, bc^2 are all strictly larger than $a^2b^2c^2$.

A1086. Let N be the largest such n. We prove that $N = 5$.

We call a set of points *allowable* if it satisfies the conditions in the statement. Note first that $N \geq 5$ because the set consisting of the four vertices and the center of a square is allowable. For the sake of seeking a contradiction, let $\mathcal{S}$ be an allowable set of $N > 5$ points. Let $\overline{\mathcal{S}}$ be the convex hull of $\mathcal{S}$. By hypothesis, $\mathcal{S}$ is not contained in a line, so $\overline{\mathcal{S}}$ is a polygon whose vertices belong to $\mathcal{S}$. Let k be the number of sides of $\overline{\mathcal{S}}$. We consider three separate cases:

Case 1: $k \geq 5$. Here three of the points are vertices of an angle of $\overline{\mathcal{S}}$ having size at least 108°, yielding a contradiction.

Case 2: $k = 4$. Unless $\overline{\mathcal{S}}$ is a rectangle $ABCD$, it has at least one obtuse angle, leading to a contradiction. Furthermore, no point P of $\mathcal{S}$ may lie on the interior of any side, say $\overline{AB}$, of $\overline{\mathcal{S}}$, for otherwise one of the angles $\angle APC$, $\angle BPC$ is obtuse yielding a contradiction. Next, if $P \in \mathcal{S}$ is an interior point of $\overline{\mathcal{S}}$, then none of the angles $\angle APB$, $\angle BPC$, $\angle CPD$, $\angle DPA$ are obtuse, so these angles must all be right angles (as their sum is 360°). This implies that the rectangle $ABCD$ is a square and P its center, contradicting the assumption that $\mathcal{S}$ has $N > 5$ points.

Case 3: $k = 3$. In this case $\overline{\mathcal{S}}$ is a triangle $\triangle ABC$. No point P of S may lie in the interior of $\triangle ABC$, for otherwise one of the angles $\angle APB$, $\angle BPC$, $\angle CPA$ would be obtuse of at least 120°. Any point P lying on the interior of a side of $\triangle ABC$ must be the foot of an altitude, for otherwise if, say, P lies on $\overline{AB}$, then one of the angles $\angle APC$, $\angle BPC$ would be obtuse contradicting the admissibility of $\mathcal{S}$. Thus, by the assumption that $\mathcal{S}$ contains $N > 5$ points, we must have that S consists of exactly $N = 6$ distinct points, namely the vertices A, B, C and the corresponding feet of altitudes A', B', C' of a necessarily acute triangle. However, in this situation angles $\angle AB'A'$, $\angle AC'A'$, $\angle BA'B'$, $\angle BC'B'$, $\angle CA'C'$ and $\angle CB'C'$ are all obtuse, a contradiction.

We conclude that $N = 5$ is the largest size of an allowable set.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Page, Scott E., *The Model Thinker: What You Need to Know to Make Data Work for You*, Basic Books, 2018; xiii + 410 pp, $32. ISBN 978-0-465-09462-2.

This book is the 30th draft of a distillation of material delivered in a massive online course to more than a million students. The book is a cornucopia of dozens of models, for all kinds of phenomena: data distributions, networks, random walks, systems dynamics, cooperation, learning, and more. Author Page emphasizes using multiple models to gain understanding of a situation, and the book concludes with applying such “many-model thinking” to the opioid epidemic and to income inequality. Page gives good prose explanations of the models (with some examples and details relegated to notes at the end of the volume), but mathematical formulations are boxed away from the text and not delved into. There are no exercises. The book could be used in a course about modeling in general for an audience with no mathematical prerequisite; for a course in mathematical modeling, it would have to be supplemented with serious investigation of the details and assumptions of the models, plus modeling projects.

Pearl, Judea, and Dana Mackenzie, *The Book of Why: The New Science of Cause and Effect*, Basic Books, 2018; x + 419 pp, $32. ISBN 978-0-465-09760-9.

Bracketed between a chapter hyping “the new science of causal inference” and one celebrating the imminence of machines that can “think” (are “self-aware” and “can distinguish good from evil, at least as reliably as humans”—!), this book contains valuable analyses of numerous experiments and networks of causation. The main tool, though scarcely new, is the causal diagram, with arrows indicating cause-effect relationships. Also not new is the distinction between an experiment (involving an intervention or treatment) and an observational study, though the authors emphasize the difference in a new notation to compare the respective conditional probabilities: $P((L|do(D))$ vs. $P(L|D)$.

Wainer, Howard, *Truth or Truthiness: Distinguishing Fact from Fiction by Learning to Think Like a Data Scientist*, Cambridge University Press, 2016; xviii + 210 pp, $29.99. ISBN 978-1-107-13057-9.

For the benefit of readers outside of contemporary U.S. culture, let me begin with a definition of “truthiness” in this context: It is the quality of being felt to be true, as based solely on one’s prior belief or intuition, without regard to facts or evidence. Author Wainer crusades on behalf of demanding evidence for a statistical claim, from either a randomized controlled experiment or post-hoc matching of naturally occurring groups in an observational study, and he concentrates on how to handle missing data. Most of his examples are drawn from education: accommodations for examinees with disabilities, teaching to the test, racial differences in testing, the economic value of teacher tenure, detecting cheating, exam subscores, and changes in the SAT. Several chapters are devoted to data presentations, and one deals with the association of earthquakes in Oklahoma with fracking for oil.

Math. Mag. **91** (2018) 396–397. doi:10.1080/0025570X.2018.1528112 © Mathematical Association of America

Hottinger, Sara N., *Inventing the Mathematician: Gender, Race, and Our Cultural Understanding of Mathematics*, State University of New York Press, 2016; ix + 205 pp, $75, $20.95(P). ISBN 978-1-4384-6009-3, 978-1-4384-6010-9.

The author connects her experiences of mathematics (as an undergraduate mathematics major) with feminist theory, investigating "where and how we get out ideas about mathematics." In particular, she details how histories of mathematics, portraits of mathematicians, mathematics textbooks, and endeavors toward ethnomathematics shape whether students can envision themselves as mathematicians. She asserts that mathematical knowledge is not value-free but instead has been "constructed in ways that limit access to select groups of people," so that "a normative mathematical subjectivity has become central to the construction of Western subjectivity and of the West itself."

Folger, Tim (September 2018). Chance encounters: How random numbers have influenced spies, scientists and reality itself, *Discover* 39(7): 54–60.

"The internet. . . could not function without random numbers. They're the foundations of online security, protecting everything from the national electric grid to the sale of airline tickets." But "the technologies that generate random numbers are straining to match the unceasing growth in internet traffic," and the system is not as secure as you might think. This article suggests moving random-number generation from mathematics (pseudorandom-number generators) to physics (the"chaotic jostling of photons"). There is already a marketed device that does that; and newly sensitive photon detectors, used by the National Institute of Standards and Technology in a test of Einstein's "local realism" view of quantum entanglement, may provide a new fount of random numbers.

Conover, Emily, Emmy Noether's vision, *Science News* (23 June 2018): 20–25.

The author briefly summarizes Noether's career and then hones in on one of her 1918 theorems, that every continuous symmetry has a corresponding conservation law, and conversely. For example, translation symmetry in time and in space correspond respectively to conservation of energy and of momentum, while rotational symmetry is connected to conservation of angular momentum. Current research in particle physics seeks particles that correspond to supersymmetry theory, which holds that each type of particle has a corresponding heavier partner particle; and Noether's theorems are also inspiring ideas about quantum gravity.

Kernighan, Brian W., *Millions, Billions, Zillions: Defending Yourself in a World of Too Many Numbers*, Princeton University Press, 2018; xiv + 158 pp, $22.95. ISBN 978-0-691-18277-3.

Quantitative reasoning requires numeracy as a base: being able to interpret and work with numbers. The skills involved include proficiency about number names, scientific notation, percentages, units of measurement, appropriate precision, graphical displays, approximation, estimation, and skepticism. This book, by a famous computer scientist, tackles all these topics with real-life examples from news sources. An accompanying workbook of exercises would make this a valuable textbook for teaching "number sense."

Krantz, Steven G., *A Primer of Mathematical Writing, Being a Disquisition on Having Your Ideas Recorded, Typeset, Published, Read and Appreciated*, 2nd ed., American Mathematical Society, 2017; xx + 243 pp, $45(P). ISBN 978-1-4704-3658-2.

In the more than 20 years since the first edition of this book, mathematical publishing has assumed many new forms: preprint servers, Web pages, blogs, wikis, and collaboration sites, not to mention electronic-only journals and their diabolical cousins, predatory journals. In addition to small emendations of Krantz's previous excellent advice about writing itself, this edition details these new outlets for mathematical creativity. Every aspiring author of mathematics should have this book. (Steve, I applaud your flexibility in appreciating some uses of passive voice, for a "certain latitude that we do not want to forfeit." But the editor in me bridles at "It may be noted that. . . " (p. 192). With best wishes, Paul C.)

Acknowledgments

In addition to our Associate Editors, the following referees have assisted the MAGAZINE *during the past year (May 2017 to May 2018). We thank them for their time and care.*

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Math. Mag. **91** (2018) 398–400. doi:10.1080/0025570X.2018.1524691.

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